



## HIGH-ORDER FINITE DIFFERENCE SCHEMES FOR SOLVING THE ADVECTION-DIFFUSION EQUATION

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**Abstract-** Up to tenth-order finite difference schemes are proposed in this paper to solve one-dimensional advection-diffusion equation. The schemes based on high-order differences are presented using Taylor series expansion. To obtain the solutions, up to tenth-order finite difference schemes in space and a fourth-order Runge-Kutta scheme in time have been combined. The methods are implemented to solve two problems having exact solutions. Numerical experiments have been conducted to demonstrate the efficiency and high-order accuracy of the current methods. The techniques are seen to be very accurate in solving the advection-diffusion equation for  $Pe \leq 5$ . The produced results are also seen to be more accurate than some available results given in the literature.

**Key Words-** Advection-Diffusion Equation, Contaminant Transport, High-order Finite Difference Schemes, Runge-Kutta.

### 1. INTRODUCTION

Advection–diffusion equation illustrates many quantities such as mass, heat, energy, velocity, vorticity, etc. The solutions of the equation model some of the phenomena such as the contaminant transport in groundwater, spread of pollutants in rivers, contaminant dispersion in shallow lakes and reservoirs etc. The slow progress has been made towards the analytical solutions of the advection-diffusion equation when initial and boundary conditions are intricate. Since many of the analytical solutions have not much easy use, many attempts have been carried out on developing the accurate numerical techniques. A number of numerical techniques have been recommended to illuminate physical phenomena described by the advection-diffusion equation in various fields of science. The difficulties arising in numerical solutions of the advection-diffusion equation results from the dominant advection that is for relatively high Peclet number. The one-dimensional advection-diffusion transport equation without the source terms has the following form:

$$c_t + uc_x - Dc_{xx} = 0, \quad a \leq x \leq b, \quad (1)$$

where the subscripts  $t$  and  $x$  stand for differentiation with respect to time and space, respectively. Also  $c$ ,  $u$  and  $D$  indicate concentration, velocity of water flow and the

diffusion coefficient, respectively. Here  $a$  and  $b$  show the physical constants. The initial condition is

$$c(x, 0) = f(x), \quad a \leq x \leq b, \quad (2)$$

and the boundary conditions are

$$c(0, t) = g_0(t), \quad 0 < t \leq T, \quad (3)$$

$$c(1, t) = g_1(t), \quad 0 < t \leq T, \quad (4)$$

where  $f$ ,  $g_0$  and  $g_1$  are prescribed functions whilst  $u$  is the unknown function, concentration. Equation (1) describes two processes: advection and diffusion. Notice that  $D > 0$  and  $u > 0$  are considered to be positive constants quantifying the diffusion and advection processes, respectively.

To solve the advection-diffusion equation with the finite difference method, Noye and Tan [1] has used a weighted discretization with the modified equivalent partial differential equation. Soon after, the authors extended this scheme to solve two-dimensional advection-diffusion equation [2]. However, solution of two- and three-dimensional problems by using these methods is difficult since requirement of matrix inversions at each time step. The upwind scheme [3] and the flux-corrected scheme [4] are available for the solution of the depth-averaged form of the advection-diffusion equation. An alternative widely used approach is the split-operator approach [5, 6], in which the advection and diffusion terms are solved by two various numerical methods. To solve the advection-diffusion equation accurately, various versions of the finite difference methods have been used in the literature [7-18]. Stability of various versions of the finite difference schemes for the advection-diffusion problems have been carried out in several studies in the literature [19-23]. The various forms of the finite difference schemes previously have been successfully described by means of using various combinations in getting the numerical solution of some problems represented by partial differential equations [24-29]. In this paper, highly accurate solutions of the advection-diffusion equation are obtained using up to tenth-order finite difference schemes in space and a fourth-order Runge-Kutta (RK4) scheme in time. The numerical computations showed that the high-order schemes produce very accurate solutions in comparison with some previous works. Furthermore, these schemes can therefore be used for solving partial differential equations encountered in various fields of science.

## 2. THE HIGH-ORDER SCHEMES

Spatial derivatives are evaluated by various orders of finite difference schemes. The spatial derivative  $c'_i$  at point  $i$  can be approximated by,  $(R + L + 1)$ -point stencil,  $(R + L)$ -order finite-difference scheme as:

$$c'_i = \frac{1}{h} \sum_{j=-L}^R a_{j+L} c_{i+j}, \quad 1 \leq i \leq N. \quad (5)$$

where  $h = x_{i+1} - x_i$  is the spacing of uniform mesh. The above formula involves  $(R + L + 1)$  constants,  $a_0, a_1, a_2, \dots, a_{R+L}$  which need to be known at point  $i$ .  $R$  and  $L$

indicates number of points in the right hand side and the left hand side for the taken stencil, respectively.  $R$  is equal to  $L$  for the considered stencil at internal points but this is not the case for the boundary nodes.  $N$  is the number of grid points. The coefficients  $a_j$  were determined with Taylor series expansion of (5). Thus, the schemes using 7, 9 and 11 points, hereafter referred to as FD6, FD8 and FD10, are of order 6, 8 and 10, respectively.

The coefficients  $a_j$  for the first derivatives in the FD6, FD8 and FD10 schemes are given at internal nodes in Table 1.

Table 1. The coefficients  $a_j$  for the first derivatives in the schemes at internal nodes

Order	$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$a_7$	$a_8$	$a_9$	$a_{10}$
6	$\frac{-1}{60}$	$\frac{9}{60}$	$\frac{-45}{60}$	0	$\frac{45}{60}$	$\frac{-9}{60}$	$\frac{1}{60}$	--	--	--	--
8	$\frac{3}{840}$	$\frac{-32}{840}$	$\frac{168}{840}$	$\frac{-672}{840}$	0	$\frac{672}{840}$	$\frac{-168}{840}$	$\frac{32}{840}$	$\frac{-3}{840}$	--	--
10	$\frac{-2}{2520}$	$\frac{25}{2520}$	$\frac{-150}{2520}$	$\frac{600}{2520}$	$\frac{-2100}{2520}$	0	$\frac{2100}{2520}$	$\frac{-600}{2520}$	$\frac{150}{2520}$	$\frac{-25}{2520}$	$\frac{2}{2520}$

Table 2a. The coefficients  $a_j$  are given for the boundary nodes (1-5) for the relevant schemes

Node	Order	$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$a_7$	$a_8$	$a_9$	$a_{10}$
1	6	$\frac{-147}{60}$	$\frac{360}{60}$	$\frac{-450}{60}$	$\frac{400}{60}$	$\frac{-225}{60}$	$\frac{72}{60}$	$\frac{-10}{60}$	--	--	--	--
	8	$\frac{-2283}{840}$	$\frac{6720}{840}$	$\frac{-11760}{840}$	$\frac{15680}{840}$	$\frac{-14700}{840}$	$\frac{9408}{840}$	$\frac{-3920}{840}$	$\frac{960}{840}$	$\frac{-105}{840}$	--	--
	10	$\frac{-7381}{2520}$	$\frac{53200}{2520}$	$\frac{-56700}{2520}$	$\frac{100800}{2520}$	$\frac{-132300}{2520}$	$\frac{127008}{2520}$	$\frac{-88200}{2520}$	$\frac{43200}{2520}$	$\frac{-14175}{2520}$	$\frac{2800}{2520}$	$\frac{-252}{2520}$
2	6	$\frac{-10}{60}$	$\frac{-77}{60}$	$\frac{150}{60}$	$\frac{-100}{60}$	$\frac{50}{60}$	$\frac{-15}{60}$	$\frac{2}{60}$	--	--	--	--
	8	$\frac{-105}{840}$	$\frac{-1338}{840}$	$\frac{2940}{840}$	$\frac{-2940}{840}$	$\frac{2450}{840}$	$\frac{-1470}{840}$	$\frac{588}{840}$	$\frac{-140}{840}$	$\frac{15}{840}$	--	--
	10	$\frac{-252}{2520}$	$\frac{-4609}{2520}$	$\frac{11340}{2520}$	$\frac{-15120}{2520}$	$\frac{17640}{2520}$	$\frac{-15876}{2520}$	$\frac{10584}{2520}$	$\frac{-5040}{2520}$	$\frac{1620}{2520}$	$\frac{-315}{2520}$	$\frac{28}{2520}$
3	6	$\frac{2}{60}$	$\frac{-24}{60}$	$\frac{-35}{60}$	$\frac{80}{60}$	$\frac{-30}{60}$	$\frac{8}{60}$	$\frac{-1}{60}$	--	--	--	--
	8	$\frac{15}{840}$	$\frac{-240}{840}$	$\frac{-798}{840}$	$\frac{1680}{840}$	$\frac{-1050}{840}$	$\frac{560}{840}$	$\frac{-210}{840}$	$\frac{48}{840}$	$\frac{-5}{840}$	--	--
	10	$\frac{28}{2520}$	$\frac{-560}{2520}$	$\frac{-3609}{2520}$	$\frac{6720}{2520}$	$\frac{-5880}{2520}$	$\frac{4704}{2520}$	$\frac{-2940}{2520}$	$\frac{1344}{2520}$	$\frac{-420}{2520}$	$\frac{80}{2520}$	$\frac{-7}{2520}$
4	8	$\frac{-5}{840}$	$\frac{60}{840}$	$\frac{-420}{840}$	$\frac{-378}{840}$	$\frac{1050}{840}$	$\frac{-420}{840}$	$\frac{140}{840}$	$\frac{-30}{840}$	$\frac{3}{840}$	--	--
	10	$\frac{-7}{2520}$	$\frac{105}{2520}$	$\frac{-945}{2520}$	$\frac{-1914}{2520}$	$\frac{4410}{2520}$	$\frac{-2646}{2520}$	$\frac{1470}{2520}$	$\frac{-630}{2520}$	$\frac{189}{2520}$	$\frac{-35}{2520}$	$\frac{3}{2520}$
5	10	$\frac{3}{2520}$	$\frac{-40}{2520}$	$\frac{270}{2520}$	$\frac{-1440}{2520}$	$\frac{-924}{2520}$	$\frac{3024}{2520}$	$\frac{-1260}{2520}$	$\frac{480}{2520}$	$\frac{-135}{2520}$	$\frac{24}{2520}$	$\frac{-2}{2520}$

Table 2b. The coefficients  $a_j$  are given for the boundary nodes ((N-4)-N) for the relevant schemes

Node	Order	$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$a_7$	$a_8$	$a_9$	$a_{10}$
N-4	10	$\frac{2}{2520}$	$\frac{-24}{2520}$	$\frac{135}{2520}$	$\frac{-480}{2520}$	$\frac{1260}{2520}$	$\frac{-3024}{2520}$	$\frac{924}{2520}$	$\frac{1440}{2520}$	$\frac{-270}{2520}$	$\frac{40}{2520}$	$\frac{-3}{2520}$
	8	$\frac{-3}{840}$	$\frac{30}{840}$	$\frac{-140}{840}$	$\frac{420}{840}$	$\frac{-1050}{840}$	$\frac{378}{840}$	$\frac{420}{840}$	$\frac{-60}{840}$	$\frac{5}{840}$	--	--
N-3	10	$\frac{-3}{2520}$	$\frac{35}{2520}$	$\frac{-189}{2520}$	$\frac{630}{2520}$	$\frac{-1470}{2520}$	$\frac{2646}{2520}$	$\frac{-4410}{2520}$	$\frac{1914}{2520}$	$\frac{945}{2520}$	$\frac{-105}{2520}$	$\frac{7}{2520}$
	6	$\frac{1}{60}$	$\frac{-8}{60}$	$\frac{30}{60}$	$\frac{-80}{60}$	$\frac{35}{60}$	$\frac{24}{60}$	$\frac{-2}{60}$	--	--	--	--
N-2	8	$\frac{5}{840}$	$\frac{-48}{840}$	$\frac{210}{840}$	$\frac{-560}{840}$	$\frac{1050}{840}$	$\frac{-1680}{840}$	$\frac{798}{840}$	$\frac{240}{840}$	$\frac{-15}{840}$	--	--
	10	$\frac{7}{2520}$	$\frac{-80}{2520}$	$\frac{420}{2520}$	$\frac{-1344}{2520}$	$\frac{2940}{2520}$	$\frac{-4704}{2520}$	$\frac{5880}{2520}$	$\frac{-6720}{2520}$	$\frac{3069}{2520}$	$\frac{560}{2520}$	$\frac{-28}{2520}$
N-1	6	$\frac{-2}{60}$	$\frac{15}{60}$	$\frac{-50}{60}$	$\frac{100}{60}$	$\frac{-150}{60}$	$\frac{77}{60}$	$\frac{10}{60}$	--	--	--	--
	8	$\frac{-15}{840}$	$\frac{140}{840}$	$\frac{-588}{840}$	$\frac{1470}{840}$	$\frac{-2450}{840}$	$\frac{2940}{840}$	$\frac{-2940}{840}$	$\frac{1338}{840}$	$\frac{105}{840}$	--	--
N	10	$\frac{-28}{2520}$	$\frac{315}{2520}$	$\frac{-1620}{2520}$	$\frac{5040}{2520}$	$\frac{-10584}{2520}$	$\frac{15876}{2520}$	$\frac{-17640}{2520}$	$\frac{15120}{2520}$	$\frac{-11340}{2520}$	$\frac{4609}{2520}$	$\frac{252}{2520}$
	6	$\frac{10}{60}$	$\frac{-72}{60}$	$\frac{225}{60}$	$\frac{-400}{60}$	$\frac{450}{60}$	$\frac{-360}{60}$	$\frac{147}{60}$	--	--	--	--
N	8	$\frac{105}{840}$	$\frac{-960}{840}$	$\frac{3920}{840}$	$\frac{-9408}{840}$	$\frac{14700}{840}$	$\frac{-15680}{840}$	$\frac{11760}{840}$	$\frac{-6720}{840}$	$\frac{2283}{840}$	--	--
	10	$\frac{252}{2520}$	$\frac{-2800}{2520}$	$\frac{14175}{2520}$	$\frac{-43200}{2520}$	$\frac{88200}{2520}$	$\frac{-127008}{2520}$	$\frac{132300}{2520}$	$\frac{-100800}{2520}$	$\frac{56700}{2520}$	$\frac{-25200}{2520}$	$\frac{7381}{2520}$

Truncation error terms for the first-order derivative of the schemes are given by:

$$TE = c'_i - \frac{1}{h} \sum_{j=-L}^R a_{j+L} c_{i+j} .$$

When the Taylor expansion of the second term in the RHS of the formula in above is used, the truncation terms of the schemes FD6, FD8, FD10 can be obtained as, respectively,  $-0.007143 h^6 c_i^{(7)}$ ,  $0.001587 h^8 c_i^{(9)}$ ,  $-0.000361 h^{10} c_i^{(11)}$ .

The coefficients for the first derivatives in the FD6, FD8 and FD10 schemes are given at boundary nodes in Table 2.

First-order spatial derivative terms can be re-written into matrix form as follows:

$$C' = AC . \tag{6}$$

The second-order spatial derivatives are obtained by applying the first-order operator twice, i.e.,

$$C'' = AC' \tag{7}$$

where  $C = (c_1, c_2, \dots, c_N)^T$ .

For the approximate solutions of equation (1) with the boundary conditions (2) using the FD methods, first the interval  $[a, b]$  is discretized such that  $a = x_1 < x_2 < \dots < x_N = b$  where  $N$  is the number of grid points. After application of the

FD techniques to equation (1), the equation can be reduced into a set of ordinary differential equations in time. Then the governing equation becomes

$$\frac{dc_i}{dt} = Pc_i \quad (8)$$

where  $P$  indicates a spatial differential operator. Each spatial derivative on the right hand side of equation (8) was computed using the present schemes and then the semi-discrete equation (8) was solved using the RK4 scheme, through the operations,

$$\begin{aligned} c_i^{(1)} &= c_i^n + \frac{1}{2} \Delta t P(c_i^n), \\ c_i^{(2)} &= c_i^n + \frac{1}{2} \Delta t P(c_i^{(1)}), \\ c_i^{(3)} &= c_i^n + \Delta t P(c_i^{(2)}), \\ c_i^{n+1} &= c_i^n + \frac{1}{6} \Delta t \left[ P(c_i^n) + 2P(c_i^{(1)}) + 2P(c_i^{(2)}) + P(c_i^{(3)}) \right]. \end{aligned} \quad (9)$$

### 3. NUMERICAL ILLUSTRATIONS

To show the performance of the methods for the model problem, various error norms are reported. Let us consider the advection-diffusion equation (1) with the initial condition (2) and boundary conditions (3-4). The results are compared with the analytical and some numerical solutions. The numerical computations were performed using uniform grids. The computed solutions are shown in Tables 3-8.

An important non-dimensional parameter in numerical analysis is the Courant( $Cr$ ) number. This parameter gives the fractional distance relative to the grid spacing travelled due to advection in a single time step  $Cr = (u\Delta t)/h$ . It is possible to show using a Fourier error analysis that for a forward difference in time approximation (i.e. explicit), no matter what approximation is used for the spatial derivatives, that the transport equation is stable for values of the  $Cr < 1$ . This stability constraint for explicit transport equations states that one cannot advect the concentration more than one grid cell in a single time step.

The Peclet number is another important non-dimensional term which compares the characteristic time for dispersion and diffusion given a length scale with the characteristic time for advection. In numerical analysis, one normally refers to a grid Peclet number  $Pe = (uh)/D$ , where  $u$  is the velocity of water flow and the characteristic length scale is given by the grid spacing  $h$ . More details on the effects of the Courant and Peclet numbers on the results can be found in [30].

Two examples for which the exact solutions are known are now used to test the methods described for solving the advection-diffusion equation. Elapsed time  $t$  is taken to be 1 s and 5 s in Example 1 and Example 2, respectively. The techniques are applied to solve (1)–(4) with  $g_0(t)$ ,  $g_1(t)$  and  $f(x)$  prescribed.

**Example 1** [17, 31]. Consider (1)–(4) with

$$f(x) = \exp\left(-\frac{(x+0.5)^2}{0.00125}\right), \quad g_0(t) = \frac{0.025}{\sqrt{0.000625 + 0.02t}} \exp\left(-\frac{(0.5-t)^2}{0.00125 + 0.04t}\right).$$

$$g_1(t) = \frac{0.025}{\sqrt{0.000625 + 0.02t}} \exp\left(-\frac{(1.5-t)^2}{0.00125 + 0.04t}\right)$$

for which the exact solution to the one-dimensional advection–diffusion in a region bounded by  $0 \leq x \leq 1$  is taken from [17] and given as:

$$c(x,t) = \frac{0.025}{\sqrt{0.000625 + 0.02t}} \exp\left(-\frac{(x+0.5-t)^2}{0.00125 + 0.04t}\right). \quad (10)$$

The parameters used are  $D = 0.01 \text{ m}^2/\text{s}$  and  $u = 1.0 \text{ m/s}$ . The produced results and exact solutions are compared in Table 3 for the various values of the Courant number and for the different grid sizes. Both quantitative and qualitative agreement between the exact and the approximate solutions is excellent (see Tables 3-5, Figures 1a,b). Figures 1a,b show numerically calculated concentration profiles using the present approximations at grid Peclet numbers of 2 and 5, respectively, for the same Courant number  $Cr = 0.2$  with various elapsed times. It has been seen that the approximated profiles are in very good agreement with their analytical counterparts.

For comparison purpose, for  $Pe = 2$ ,  $Cr = 0.2$ ,  $\Delta t = 0.004$ ,  $h = 0.02$  at  $x = 0.5$ , absolute errors have also been computed and seen to be  $4.03\text{E-}07$ ,  $1.42\text{E-}09$  and  $6.88\text{E-}09$  for the FD6, FD8 and FD10 respectively. It can be seen that these results are far better comparison to Dehghan's result [17] which is  $1.80\text{E-}06$ . Note that the same problem is solved in [16, 17] by using various forms of finite difference schemes. In the present model, using the present schemes, more accurate results are obtained in comparison with the corresponding references. As seen in Table 4, the high-order FD results are also compared with the results of Kadalbajoo and Arora [31] used the Taylor-Galerkin B-spline finite element methods. It is seen that the present results are more accurate than their results. However, in order to improve the accuracy, the present work uses smaller time increment comparison to them.

The results of the three schemes for the equation are compared in Table 6 for this example. Table 5 shows that the accuracy of the solutions improves rapidly as the mesh size is reduced. All comparisons show that among the proposed schemes, the FD10 usually offer better results than the rest of two schemes. Therefore for the coming example, only the FD10 scheme is considered to observe the physical and mathematical behavior of the problem.

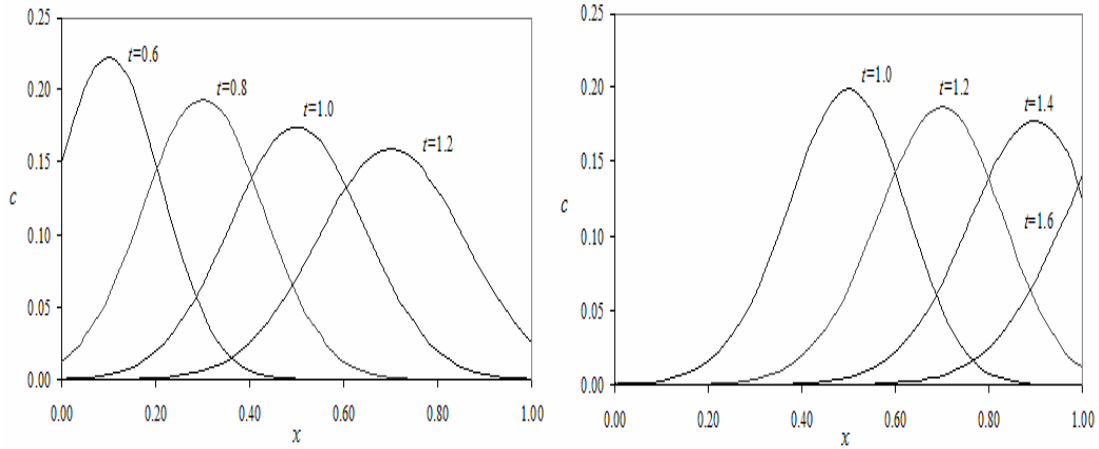


Figure 1. FD10 solution at various times in Example 1 for (a)  $h=0.02$ ,  $0 \leq x \leq 1$ ,  $\Delta t=0.004$ ,  $u=1.0$  m/s,  $D=0.01$  m<sup>2</sup>/s,  $Cr=0.2$ ,  $Pe=2$ , (b)  $h=0.025$ ,  $0 \leq x \leq 1$ ,  $\Delta t=0.01$ ,  $u=1.0$  m/s,  $D=0.005$  m<sup>2</sup>/s,  $Cr=0.4$ ,  $Pe=5$ .

**Example 2** [13]. Let us consider equation (1) for  $3.5 \leq x \leq 6.5$ ,  $u=0.8$  m/s and  $D=0.005$  m<sup>2</sup>/s.

Then the exact solution to equation (1) as given by Noye and Tan [1] is

$$c(x,t) = \frac{1}{\sqrt{4t+1}} \exp\left[-\frac{(x-1-ut)^2}{D(4t+1)}\right]. \tag{11}$$

The initial and boundary conditions (2)-(4) are taken from the exact solution. The distribution of the Gaussian pulse is computed using the FD10 solutions as shown in Figure 2. As can be seen in Table 7 that the FD10 results in Example 2 is far more accurate comparison to Karahan’s results [16]. In Table 4, various error norms of FD10 solutions for various values of  $h$ ,  $Pe$ ,  $\Delta t$ ,  $Cr$  with  $x=5$ ,  $t=5$  have also been presented in Example 2. FD10 solutions in addition to absolute and relative errors have also presented in Table 8 for various values of  $x$  with  $h=0.025$ ,  $\Delta t=0.005$ ,  $Cr=0.16$  and  $Pe=4$ .

#### 4. CONCLUSIONS

In this work the advection-diffusion processes were dealt with using up to tenth-order finite difference schemes in space and the RK4 in time. The methods successfully worked to give very accurate solutions to these processes for  $Pe \leq 5$ . The performance of the schemes for the considered problems was measured by calculating the various error indicators. The methods give convergent approximations, and handle the advection-diffusion problems. The schemes based on high-order differences provide efficient and alternative methods for modelling the behaviour of the problem. Comparisons of the results with exact solutions showed that the present schemes are capable of solving the. For a further research, one can concentrate on solving the advection-diffusion problems using high Peclet numbers,  $Pe > 5$ , with high-order upwind schemes. Restrictions of the stability of the high-order FD schemes for the advection-diffusion problems will be analyzed in a future study in detail.

Table 3. Numerical solutions for different schemes with  $x = 0.5$ ,  $t = 1.0$ ,  $Pe = 2$  in Example 1.

$\Delta t$	$h = 0.02, Pe = 2$					$h = 0.01, Pe = 1$				
	$Cr$	Exact	FD6	FD8	FD10	$Cr$	Exact	FD6	FD8	FD10
0.0005	0.025	0.17407766	0.17407682	0.17407772	0.17407764	0.050	0.17407766	0.17407765	0.17407766	0.17407766
0.0010	0.050	0.17407766	0.17407689	0.17407772	0.17407765	0.100	0.17407766	0.17407765	0.17407766	0.17407766
0.0020	0.100	0.17407766	0.17407702	0.17407769	0.17407765	0.200	0.17407766	0.17407765	0.17407766	0.17407766
0.0033	0.167	0.17407766	0.17408557	0.17408607	0.17408605	0.333	0.17407766	---	---	---
0.0040	0.200	0.17407766	0.17407725	0.17407766	0.17407765	0.400	0.17407766	---	---	---
0.0080	0.400	0.17407766	0.17407757	0.17407761	0.17480605	0.800	0.17407766	---	---	---
0.0100	0.500	0.17407766	0.17407766	---	---	1.000	0.17407766	---	---	---

Table 4. A comparison of present solutions with the results of Kadalbajoo and Arora [31] at various nodal points with  $h = 0.01$ ,  $t = 1.0$  and  $Pe = 1$  in Example 1.

$x$	Exact	$\Delta t = 0.001, Cr=0.1$			$\Delta t = 0.01, Cr=1$					
		FD6	FD8	FD10	TGBS1Q [31]	TGBS1L [31]	TGBS2Q [31]	TGBS2L [31]	TGBS2HQ [31]	TGBS2HL [31]
0.1	0.0035992	0.0035992	0.0035992	0.0035992	0.0035860	0.0035872	0.0036345	0.0036360	0.0036055	0.0036070
0.2	0.0196422	0.0196423	0.0196423	0.0196423	0.0196139	0.0196087	0.0196514	0.0196436	0.0196048	0.0195969
0.3	0.0660098	0.0660099	0.0660099	0.0660099	0.0660910	0.0660729	0.0656778	0.0656522	0.0658887	0.0658624
0.4	0.1366027	0.1366028	0.1366028	0.1366028	0.1368574	0.1368703	0.1361951	0.1362141	0.1366971	0.1367171
0.5	0.1740776	0.1740777	0.1740777	0.1740777	0.1740807	0.1741277	0.1745342	0.1746041	0.1744042	0.1744743
0.6	0.1366027	0.1366028	0.1366028	0.1366028	0.1362195	0.1362113	0.1373380	0.1373258	0.1365433	0.1365294
0.7	0.0660098	0.0660099	0.0660099	0.0660099	0.0658153	0.0657761	0.0659965	0.0659361	0.0657310	0.0656721
0.8	0.0196422	0.0196423	0.0196423	0.0196423	0.0197571	0.0197514	0.0193274	0.0193197	0.0196024	0.0195950
0.9	0.0035992	0.0035992	0.0035992	0.0035992	0.0037190	0.0037288	0.0034599	0.0034754	0.0036699	0.0036847



Table 5. Different error norms for various values of  $Cr$  and  $\Delta t$  with ( $x = 0.5$  for absolute and relative errors),  $t = 1.0$ ,  $Pe = 2$  in Example 1.

Scheme	$\Delta t$	$h = 0.02, Pe = 2$					$h = 0.01, Pe = 1$				
		$Cr$	Absolute error	Relative error	$L_2$	$L_\infty$	$Cr$	Absolute error	Relative error	$L_2$	$L_\infty$
FD6	0.0005	0.025	8.40E-07	4.83E-06	6.40E-07	1.48E-06	0.050	6.66E-09	3.82E-08	8.42E-09	2.02E-08
	0.0010	0.050	7.68E-07	4.41E-06	6.08E-07	1.42E-06	0.100	5.80E-09	3.33E-08	7.71E-09	1.86E-08
	0.0020	0.100	6.34E-07	3.64E-06	5.51E-07	1.31E-06	0.200	4.45E-09	2.56E-08	6.60E-08	1.61E-08
	0.0033	0.167	7.92E-06	4.55E-05	4.31E-05	7.43E-05	0.333	9.09E-06	5.22E-05	----	----
	0.0040	0.200	4.03E-07	2.32E-06	4.57E-07	1.11E-06	0.400	----	----	----	----
	0.0080	0.400	8.62E-08	4.95E-07	3.46E-07	8.58E-07	0.800	----	----	----	----
	0.0100	0.500	7.37E-09	4.23E-08	4.76E-05	3.33E-04	1.000	----	----	----	----
FD8	0.0005	0.025	7.37E-08	4.23E-07	6.68E-08	8.25E-08	0.050	2.42E-11	1.39E-10	3.29E-11	7.58E-11
	0.0010	0.050	6.15E-08	3.53E-07	3.17E-08	7.08E-08	0.100	2.18E-11	1.25E-10	4.71E-11	1.16E-10
	0.0020	0.100	3.90E-08	2.24E-07	2.27E-08	4.94E-08	0.200	8.51E-11	4.89E-10	8.03E-11	1.91E-10
	0.0033	0.167	8.41E-06	4.83E-05	4.30E-05	7.37E-05	0.333	----	----	----	----
	0.0040	0.200	1.42E-09	8.17E-09	1.32E-08	3.12E-08	0.400	----	----	----	----
	0.0080	0.400	4.13E-08	3.37E-07	2.20E-08	4.88E-08	0.800	----	----	----	----
	0.0100	0.500	----	----	----	----	1.000	----	----	----	----
FD10	0.0005	0.025	1.18E-08	6.76E-08	6.67E-09	1.51E-08	0.050	4.50E-12	2.58E-11	2.39E-12	5.40E-12
	0.0010	0.050	1.00E-08	5.77E-08	5.63E-09	1.27E-08	0.100	1.74E-12	9.97E-12	1.09E-12	2.49E-12
	0.0020	0.100	6.88E-09	3.95E-08	3.76E-09	8.49E-09	0.200	5.15E-12	2.96E-11	2.46E-12	5.83E-12
	0.0033	0.167	8.40E-06	4.82E-05	4.30E-05	7.37E-05	0.333	----	----	----	----
	0.0040	0.200	1.58E-09	9.08E-09	7.90E-10	1.61E-09	0.400	----	----	----	----
	0.0080	0.400	5.72E-03	3.28E-02	----	----	0.800	----	----	----	----
	0.0100	0.500	----	----	----	----	1.000	----	----	----	----

Table 6. A comparison of analytical and numerical solutions for various values of  $x$  with  $h = 0.02$ ,  $\Delta t = 0.004$ ,  $Cr = 0.2$  and  $Pe = 2$  in Example 1.

$x$	Exact	FD6		FD8		FD10	
		Results	Absolute errors	Results	Absolute errors	Results	Absolute errors
0.10	0.00359920727	0.00359921813	1.09E-08	0.00359920589	1.38E-09	0.00359920768	4.05E-10
0.20	0.01964225900	0.01964209009	1.69E-07	0.01964226493	5.93E-09	0.01964225790	1.10E-09
0.30	0.06600988665	0.06600982836	5.83E-08	0.06600987396	1.27E-08	0.06600988736	7.14E-10
0.40	0.13660277419	0.13660355137	7.78E-07	0.13660278944	1.52E-08	0.13660277500	8.09E-10
0.50	0.17407765596	0.17407725278	4.03E-07	0.17407765738	1.42E-09	0.17407765437	1.58E-09
0.60	0.13660277419	0.13660192973	8.44E-07	0.13660274865	2.55E-08	0.13660277488	6.82E-10
0.70	0.06600988665	0.06601059056	7.04E-07	0.06600991054	2.39E-08	0.06600988717	5.19E-10
0.80	0.01964225900	0.01964248474	2.26E-07	0.01964225650	2.49E-09	0.01964225832	6.73E-10
0.90	0.00359920727	0.00359895542	2.52E-07	0.00359920059	6.68E-09	0.00359920743	1.54E-10

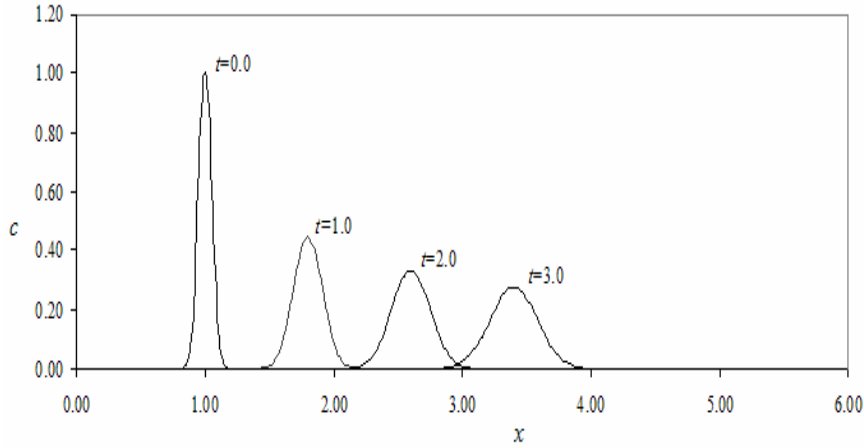


Figure 2. FD10 solutions for transport of one-dimensional Gaussian pulse at various times with the parameters  $h = 0.025$ ,  $0 \leq x \leq 6$ ,  $\Delta t = 0.005$ ,  $u = 0.8$  m/s,  $D = 0.005$  m<sup>2</sup>/s,  $Cr = 0.16$ ,  $Pe = 4$  in Example 2.

Table 7. Different error values of FD10 solutions for various values of  $h$ ,  $Pe$ ,  $\Delta t$ ,  $Cr$  with  $x = 5$ ,  $t = 5$  in Example 2.

$h, Pe$	$\Delta t$	Exact	$Cr$	FD10	Absolute error		Relative error	$L_2$	$L_\infty$
					Ref.[16]	Present			
$h=0.025,$ $Pe=4$	0.0005	0.21821789	0.016	0.21821789	9.13E-04	1.74E-09	7.97E-09	5.78E-10	1.75E-09
	0.0010	0.21821789	0.032	0.21821789	9.31E-04	1.64E-09	7.50E-09	5.44E-10	1.64E-09
	0.0020	0.21821789	0.064	0.21821789	9.73E-04	1.44E-09	6.60E-09	4.79E-10	1.44E-09
	0.0025	0.21821789	0.080	0.21821789	9.96E-04	1.35E-09	6.17E-09	4.48E-10	1.35E-09
	0.0050	0.21821789	0.160	0.21821789	1.13E-03	8.83E-10	4.05E-09	3.00E-10	9.10E-10
	0.0100	0.21821789	0.320	0.21821789	1.47E-03	7.55E-10	3.46E-09	2.58E-10	7.55E-10
	0.0200	0.21821789	0.640	----	2.26E-03	----	----	----	----
$h=0.0125,$ $Pe=2$	0.0005	0.21821789	0.032	0.21821789	8.59E-05	6.06E-13	2.78E-12	3.00E-13	9.13E-13
	0.0010	0.21821789	0.064	0.21821789	9.53E-05	4.48E-13	2.05E-12	2.64E-13	8.57E-13
	0.0020	0.21821789	0.128	0.21821789	1.16E-04	3.62E-14	1.66E-13	1.56E-13	5.72E-13
	0.0025	0.21821789	0.160	0.21821789	1.26E-04	7.51E-13	3.44E-12	2.65E-13	8.08E-13
	0.0050	0.21821789	0.320	0.21821789	1.85E-04	3.09E-11	1.41E-10	1.02E-11	3.23E-11
	0.0100	0.21821789	0.640	----	3.27E-04	----	----	----	----
	0.0200	0.21821789	1.280	----	0.217533	----	----	----	----

Table 8. FD10 solutions in addition to absolute and relative errors for various values of  $x$  with  $h = 0.025$ ,  $\Delta t = 0.005$ ,  $Cr = 0.16$  and  $Pe = 4$ .

$x$	Exact	FD10	Absolute errors	Relative errors
3.50	0.0000000	0.0000000	3.13E-09	1.60E-12
4.00	0.0000159	0.0000159	5.82E-09	4.55E-07
4.50	0.0201770	0.0201770	1.56E-08	1.32E-08
5.00	0.2182179	0.2182179	2.31E-08	4.05E-09
5.50	0.0201770	0.0201770	2.89E-08	9.56E-09
6.00	0.0000159	0.0000159	1.00E-09	4.77E-07
6.50	0.0000000	0.0000000	5.22E-13	1.30E-12

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