Filomat 28:10 (2014), 2163–2173
DOI 10.2298/FIL1410163Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence of Positive Solutions for Second-Order Impulsive Time Scale Boundary Value Problems on Infinite Intervals

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Abstract. In this paper, we consider nonlinear second order m-point impulsive time scale boundary value problems on infinite intervals. By using Leray-Schauder fixed point theorem, Avery-Henderson fixed point theorem and the five functional fixed point theorem, respectively, we establish the criteria for the existence of at least one, two and three positive solutions to the nonlinear impulsive time scale boundary value problems on infinite intervals.

1. Introduction

Impulsive problems describe processes which experience a sudden change in their states at certain moments. Impulsive differential equations have been developed in modeling impulsive problems in physics, chemical technology, population dynamics, ecology, biological systems, biotechnology, industrial robotics, optimal control, economics, and so forth. For the introduction of the theory of impulsive differential equations, we refer to the books [3–5]. Especially, the study of impulsive dynamic equations on time scales has also attracted much attention since it provides an unifying structure for differential equations in the continuous cases and finite difference equations in the discrete cases, see [6–8, 11–13, 15–17] and references therein. Some basic definitions and theorems on time scales can be found in the books [9, 10]. In recent years, there are a few authors studied the existence of positive solutions for time scale boundary value problems on infinite intervals. We refer the reader to [14, 18, 19]. Due to the fact that an infinite interval is noncompact, the discussion about boundary value problem on the infinite intervals is more complicated. To the authors knowledge, no one has studied the existence of positive solutions for m-point impulsive time scale boundary value problems for an increasing homeomorphism and positive homomorphism on infinite intervals. The results are even new for the difference equations and differential equations as well as for dynamic equations on general time scales.

We consider the following boundary value problem (BVP)

$$\begin{cases} (\varphi(y^{\Delta}(t))^{\nabla} + h(t)f\left(t, y(t), y^{\Delta}(t)\right) = 0, \quad t \in [a,\infty)_{\mathbb{T}}, \quad t \neq t_k, \quad k = 1,2,...,n \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \quad k = 1,2,...,n \\ y(a) - \beta y^{\Delta}(a) = \sum_{i=1}^{m-2} \alpha_i y^{\Delta}(\eta_i), \quad \lim_{t\to\infty} y^{\Delta}(t) = 0, \quad m \geq 3 \end{cases} \tag{1}$$

2010 *Mathematics Subject Classification*. Primary 34B18; Secondary 34B37, 34N05
Keywords. boundary value problems, cone, fixed point theorems, impulsive dynamic equations, positive solutions, time scales.
Received: 29 September 2013; Accepted: 07 October 2014
Communicated by Jelena Manojlovic
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where $\mathbb{T}$ is a time scale, $\beta \geq 0$, $\alpha_i \geq 0$ $(1 \leq i \leq m-2)$, $0 \leq a < \eta_1 < \ldots < \eta_{m-2} < \infty$, $f \in C([a,\infty)_{\mathbb{T}} \times [0,\infty) \times [0,\infty), [0,\infty))$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism and positive homomorphism with $\varphi(0) = 0$.

We will assume that the following conditions are satisfied:

(H1) $h \in C([a,\infty)_{\mathbb{T}}, [0,\infty))$, $\int\limits_a^\infty h(s)\nabla s < \infty$, $\int\limits_a^\infty \varphi^{-1}\left(\int\limits_\xi^\infty h(s)\nabla s\right)\Delta\xi < \infty$;

(H2) $f(t,(1+t)u,v) \leq \omega(\max\{|u|,|v|\})$ with $\omega \in C([0,\infty),[0,\infty))$ nondecreasing;

(H3) $\sum\limits_{a<t_k<\infty} I_k(y(t_k)) < \infty$, $I_k \in C(\mathbb{R},\mathbb{R}^+)$, $t_k \in [a,\infty)_{\mathbb{T}}$ and $y(t_k^+) = \lim\limits_{h\to 0} y(t_k+h)$, $y(t_k^-) = \lim\limits_{h\to 0} y(t_k-h)$ represent the right and left limits of $y(t)$ at $t = t_k$, $k = 1,...,n$.

The rest of paper is organized as follows. In Section 2, we give several lemmas to prove the main results in this paper. In Section 3, firstly, Leray-Schauder fixed-point theorem is used to investigate the existence of at least one positive solution of the BVP (1). Second, we apply the Avery-Henderson fixed point theorem to prove the existence of at least two positive solutions to the BVP (1). Finally, we use the five functionals fixed-point theorem to show that the existence of at least three positive solutions for the BVP (1).

2. Preliminaries

We now state and prove several lemmas which are needed later.

Lemma 2.1. *Assume (H3) holds. Let $x \in C([a,\infty)_{\mathbb{T}}, [0,\infty))$ and $\int\limits_a^\infty x(t)\nabla t < \infty$ then $y(t)$ is a solution of the following BVP*

$$\begin{cases} (\varphi\left(y^\Delta(t)\right)^\nabla + x(t) = 0, \ \ t \in [a,\infty)_{\mathbb{T}}, \ \ t \neq t_k, \ \ k = 1,2,...,n \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k)), \ \ k = 1,2,...,n \\ y(a) - \beta y^\Delta(a) = \sum\limits_{i=1}^{m-2} \alpha_i y^\Delta(\eta_i), \ \ \lim\limits_{t\to\infty} y^\Delta(t) = 0, \ \ m \geq 3 \end{cases} \tag{2}$$

if and only if $y(t)$ is a solution of the following integral equation

$$y(t) = \sum_{i=1}^{m-2} \alpha_i \varphi^{-1}\left(\int\limits_{\eta_i}^\infty x(s)\nabla s\right) + \beta\varphi^{-1}\left(\int\limits_a^\infty x(s)\nabla s\right) + \int\limits_a^t \varphi^{-1}\left(\int\limits_\xi^\infty x(s)\nabla s\right)\Delta\xi + \sum_{a<t_k<t} I_k(y(t_k)) \tag{3}$$

and $y(t) \geq 0$ for $t \in [a,\infty)_{\mathbb{T}}$.

Proof. First, suppose that $y(t)$ is a solution of problem (2). Then

$$-\left(\varphi\left(y^\Delta(t)\right)\right)^\nabla = x(t)$$

for $t \in [a,\infty)_{\mathbb{T}}$. An integration from t to ∞ of both sides of the above equality yields

$$\varphi\left(y^\Delta(t)\right) - \lim_{t\to\infty}\varphi\left(y^\Delta(t)\right) = \int\limits_t^\infty x(s)\nabla s.$$

Since φ is continuous and $\varphi(0) = 0$, we have

$$\varphi\left(y^\Delta(t)\right) = \int\limits_t^\infty x(s)\nabla s,$$

$$y^{\Delta}(t) = \varphi^{-1}\left(\int_t^{\infty} x(s)\nabla s\right). \tag{4}$$

Integrating the above equality from a to t, we get

$$y(t) - y(a) - \sum_{a<t_k<t} I_k\left(y(t_k)\right) = \int_a^t \varphi^{-1}\left(\int_{\xi}^{\infty} x(s)\nabla s\right)\Delta\xi,$$

$$y(t) = \sum_{i=1}^{m-2} \alpha_i y^{\Delta}(\eta_i) + \beta y^{\Delta}(a) + \int_a^t \varphi^{-1}\left(\int_{\xi}^{\infty} x(s)\nabla s\right)\Delta\xi + \sum_{a<t_k<t} I_k\left(y(t_k)\right).$$

Therefore, by (4), we obtain (3).

Conversely, let $y(t)$ be as in (3). Taking the delta derivative of $y(t)$ gives

$$y^{\Delta}(t) = \varphi^{-1}\left(\int_t^{\infty} x(s)\nabla s\right),$$

$$\text{i.e. } \varphi\left(y^{\Delta}(t)\right) = \int_t^{\infty} x(s)\nabla s.$$

It is easy to see that $y(t)$ satisfy (2). Furthermore, from $\beta \geq 0$, $\alpha_i \geq 0$ $(1 \leq i \leq m-2)$, $x \in C\left([a,\infty)_{\mathbb{T}}, [0,\infty)\right)$, $\int_a^{\infty} x(t)\nabla t < \infty$ and (H3), it is clear that $y(t) \geq 0$ for $t \in [a,\infty)_{\mathbb{T}}$. So, the proof of lemma is completed. $\square$

By Lemma 2.1, the solutions of the BVP (1) are the fixed points of the operator A defined by

$$\begin{aligned} Ay(t) &= \sum_{i=1}^{m-2} \alpha_i \varphi^{-1}\left(\int_{\eta_i}^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\right) + \beta\varphi^{-1}\left(\int_a^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\right) \\ &+ \int_a^t \varphi^{-1}\left(\int_{\xi}^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\right)\Delta\xi + \sum_{a<t_k<t} I_k(y(t_k)). \end{aligned}$$

Let $\mathcal{B}$ be the Banach space defined by

$$\mathcal{B} = \left\{ y \in C^{\Delta}\left([a,\infty)\right) : \sup_{t\in[a,\infty)_{\mathbb{T}}} \frac{y(t)}{1+t} < \infty, \quad \lim_{t\to\infty} y^{\Delta}(t) = 0 \right\} \tag{5}$$

with the norm $\|y\| = \max\left\{\|y\|_1, \|y^{\Delta}\|_{\infty}\right\}$, where

$$\|y\|_1 = \sup_{t\in[a,\infty)_{\mathbb{T}}} \frac{|y(t)|}{1+t}, \qquad \|y^{\Delta}\|_{\infty} = \sup_{t\in[a,\infty)_{\mathbb{T}}} |y^{\Delta}(t)|$$

and define the cone $P \subset \mathcal{B}$ by

$$\begin{aligned} P &= \Big\{ y \in \mathcal{B} : y(a) - \beta y^{\Delta}(a) = \sum_{i=1}^{m-2} \alpha_i y^{\Delta}(\eta_i), \ \ y \text{ is concave, non-decreasing and} \\ &\qquad \text{nonnegative on } [a,\infty)_{\mathbb{T}} \Big\}. \end{aligned} \tag{6}$$

Lemma 2.2. *If $y \in P$, then we have $\|y\|_1 \leq M\|y^\Delta\|_\infty$, where*

$$M = \max\left\{\beta - a + \sum_{i=1}^{m-2} \alpha_i, 1\right\}. \tag{7}$$

Proof. For $y \in P$ and $t \in [a, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
\frac{y(t)}{1+t} &= \frac{1}{1+t}\left(\int_a^t y^\Delta(s)\Delta s + \beta y^\Delta(a) + \sum_{i=1}^{m-2} \alpha_i y^\Delta(\eta_i)\right) \leq \frac{t - a + \beta + \sum\limits_{i=1}^{m-2} \alpha_i}{1+t}\|y^\Delta\|_\infty \\
&\leq M\|y^\Delta\|_\infty.
\end{aligned}$$

Hence, the proof is complete.
□

Lemma 2.3. *If (H1)-(H3) hold, then the operator $A : P \to P$ is completely continuous.*

Proof. We divide the proof into three steps.
Step 1: We show that $A : P \to P$.
In fact, for $y \in P$, we have
$(Ay)^\Delta(\infty) = 0,$
$(\varphi((Ay)^\Delta))^\nabla(t) = -h(t)f\left(t, y(t), y^\Delta(t)\right) \leq 0,$
$(Ay)^\Delta(t) = \varphi^{-1}\left(\int\limits_t^\infty h(s)f\left(s, y(s), y^\Delta(s)\right)\nabla s\right) \geq 0,$
$(Ay)(a) = \sum\limits_{i=1}^{m-2} \alpha_i(Ay)^\Delta(\eta_i) + \beta(Ay)^\Delta(a) \geq 0.$
Hence $A : P \to P$.
Step 2: We show that $A : P \to P$ is continuous.
If $y_n \to y$ as $n \to \infty$ in P, then there exists τ such that $\sup\limits_{n\in\mathbb{N}} \|y_n\| < \tau$. From (H2), for all $t \in [a, \infty)_{\mathbb{T}}$ we have $f\left(t, y_n(t), y_n^\Delta(t)\right) \leq \omega\left(\max\left\{\frac{|y_n(t)|}{1+t}, |y_n^\Delta(t)|\right\}\right) \leq \omega\left(\|y_n\|\right) < \omega(\tau)$ and $f\left(t, y(t), y^\Delta(t)\right) \leq \omega\left(\|y\|\right) < \omega(\tau)$ by the continuity of norm function. Since

$$\int_t^\infty h(s)|f\left(s, y_n(s), y_n^\Delta(s)\right) - f\left(s, y(s), y^\Delta(s)\right)|\nabla s \leq 2\omega(\tau)\int_a^\infty h(s)\nabla s < \infty$$

by using (H1), we obtain

$$\begin{aligned}
\left|\varphi\left((Ay_n)^\Delta(t)\right) - \varphi\left((Ay)^\Delta(t)\right)\right| &\leq \int_t^\infty h(s)|f\left(s, y_n(s), y_n^\Delta(s)\right) - f\left(s, y(s), y^\Delta(s)\right)|\nabla s \\
&\to 0, \ n \to \infty
\end{aligned}$$

by the Lebesgue dominated convergence theorem. Hence, we get $\|(Ay_n)^\Delta - (Ay)^\Delta\|_\infty \to 0$, as $n \to \infty$. Since $\|Ay_n - Ay\| \leq M\|(Ay_n)^\Delta - (Ay)^\Delta\|_\infty \to 0$, $A : P \to P$ is continuous.
Step 3: We show that the image of any bounded subset of P under A is relatively compact in P.
If Ω is any bounded subset of P, then there exists $K > 0$ such that $\|y\| \leq K$ for $\forall y \in \Omega$. By (H1) and (H2), for $\forall y \in \Omega$, we have

$$\|(Ay)^\Delta\|_\infty = \varphi^{-1}\left(\int_a^\infty h(s)f\left(s, y(s), y^\Delta(s)\right)\nabla s\right) \leq \varphi^{-1}\left(\omega(K)\right)\varphi^{-1}\left(\int_a^\infty h(s)\nabla s\right) < \infty.$$

Since $\|A\Omega\| \leq M\|(A\Omega)^{\Delta}\|_{\infty} < \infty$, $A\Omega$ is uniformly bounded.

Now, we show that $A\Omega$ is equicontinuous on $[a,\infty)_{\mathbb{T}}$. For any $R > 0$, $t, p \in [a,R]_{\mathbb{T}}$, and for all $y \in \Omega$, without loss of generality we may assume that $t < p$. By (H2), we have

$$\left|\varphi\left((Ay)^{\Delta}(t)\right) - \varphi\left((Ay)^{\Delta}(p)\right)\right| = \left|\int_t^p h(s)f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\right| \leq \omega(K)\int_t^p h(s)\nabla s \to 0,$$

uniformly as $t \to p$. Since $\|(Ay)^{\Delta}(t) - (Ay)^{\Delta}(p)\|_{\infty} \to 0$, uniformly as $t \to p$, we obtain $\|(Ay)(t) - (Ay)(p)\| \to 0$, uniformly as $t \to p$, by Lemma 2.2. Hence $A\Omega$ is equicontinuous on any compact interval of $[a,\infty)_{\mathbb{T}}$.

Now, we show that $A\Omega$ is equiconvergent on $[a,\infty)_{\mathbb{T}}$. For any $y \in \Omega$, by using (H1)-(H3), we have

$$\begin{aligned}\lim_{t\to\infty}\left|\frac{(Ay)(t)}{1+t}\right| &= \lim_{t\to\infty}\frac{\left(\int_a^t \varphi^{-1}\left(\int_{\xi}^{\infty} h(s)f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\right)\Delta\xi + \sum_{a<t_k<t} I_k(y(t_k))\right)}{1+t}\\ &\leq \varphi^{-1}\left(\omega(K)\right)\int_a^{\infty}\varphi^{-1}\left(\int_{\xi}^{\infty} h(s)\nabla s\right)\Delta\xi \lim_{t\to\infty}\frac{1}{1+t} = 0\end{aligned}$$

and

$$\lim_{t\to\infty}(Ay)^{\Delta}(t) \leq \varphi^{-1}\left(\omega(K)\right)\lim_{t\to\infty}\varphi^{-1}\left(\int_t^{\infty} h(s)\nabla s\right) = 0.$$

Since $\|(Ay)^{\Delta}(t) - (Ay)^{\Delta}(\infty)\|_{\infty} \to 0$, as $t \to \infty$, we obtain $\|(Ay)(t) - (Ay)(\infty)\| \to 0$, as $t \to \infty$, by Lemma 2.2. Therefore $A\Omega$ is equiconvergent on $[a,\infty)_{\mathbb{T}}$.

From steps $1-3$, the operator $A : P \to P$ is completely continuous.
□

3. Existence of Positive Solutions

To prove the existence of at least one positive solution for the BVP (1), we will apply the following Leray-Schauder Fixed Point Theorem.

Theorem 3.1. *Let E be a Banach space, $A : E \to E$ is a completely continuous operator. If the set $\{x \in E : x = \lambda Ax, 0 < \lambda < 1\}$ is bounded, then A has at least one fixed point in the closed set $T \subset E$, where*

$$T = \{x \in E : \|x\| \leq R\}, \;\; R = sup\{\|x\| : x = \lambda Ax, 0 < \lambda < 1\}.$$

Theorem 3.2. *If (H1)-(H3) hold, then the BVP* (1) *has at least one positive solution.*

Proof. Define the cone P as in (6). From Lemma 2.3, $A : P \to P$ is completely continuous.

Since $\|y\| = \max\left\{\|y\|_1, \|y^{\Delta}\|_{\infty}\right\}$, we have $\|y\| = \|y\|_1$ or $\|y\| = \|y^{\Delta}\|_{\infty}$. If $\|y\| = \|y\|_1$, then we get $\|y\| \leq M\|y^{\Delta}\|_{\infty}$ from Lemma 2.2. If $\|y\| = \|y^{\Delta}\|_{\infty}$, then we find $\|y\| \leq M\|y^{\Delta}\|_{\infty}$ by using $M \geq 1$. Hence, we obtain

$$\|y\| \leq M\|y^{\Delta}\|_{\infty} \tag{8}$$

for all $y \in P$.

We denote

$$N(A) := \{y \in P : y = \lambda Ay, 0 < \lambda < 1\}.$$

Now we show that the set $N(A)$ is bounded. Let $T = \{y \in P : \|y\| \leq R\}$ and $R = sup\{\|y\| : y = \lambda Ay, 0 < \lambda < 1\}$. Then for all $y \in N(A)$, we have

$$\begin{aligned} \|y\| &\leq M\lambda\|(Ay)^{\Delta}\|_{\infty} \\ &\leq M\lambda\varphi^{-1}\left(\int_a^{\infty} h(s)f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\right) \\ &\leq M\lambda\varphi^{-1}\left(\omega(R)\right)\varphi^{-1}\left(\int_a^{\infty} h(s)\nabla s\right) < \infty \end{aligned}$$

by (8), (H1) and (H2).

Then we obtain $N(A)$ is bounded. By Theorem 3.1, the BVP (1) has at least one positive solution.
□

We will need also the following (Avery-Henderson) fixed point theorem [1] to prove the existence of at least two positive solutions for the BVP (1).

Theorem 3.3. *[1] Let P be a cone in a real Banach space E. Set*

$$P(\phi, r) = \{u \in P : \phi(u) < r\}.$$

If η and ϕ are increasing, nonnegative continuous functionals on P, let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some positive constants r and M,

$$\phi(u) \leq \theta(u) \leq \eta(u) \text{ and } \|u\| \leq M\phi(u)$$

for all $u \in \overline{P(\phi, r)}$. Suppose that there exist positive numbers $p < q < r$ such that

$$\theta(\lambda u) \leq \lambda\theta(u), \text{ for all } 0 \leq \lambda \leq 1 \text{ and } u \in \partial P(\theta, q).$$

If $A : \overline{P(\phi, r)} \to P$ is a completely continuous operator satisfying
(i) $\phi(Au) > r$ for all $u \in \partial P(\phi, r)$,
(ii) $\theta(Au) < q$ for all $u \in \partial P(\theta, q)$,
(iii) $P(\eta, p) \neq \emptyset$ and $\eta(Au) > p$ for all $u \in \partial P(\eta, p)$,
then A has at least two fixed points u_1 and u_2 such that

$$p < \eta(u_1) \text{ with } \theta(u_1) < q \text{ and } q < \theta(u_2) \text{ with } \phi(u_2) < r.$$

Define the constant

$$N := \left(\varphi^{-1}\left(\int_a^{\infty} h(s)\nabla s\right)\right)^{-1}. \tag{9}$$

Theorem 3.4. *Assume (H1)-(H3) hold. Suppose there exist numbers $0 < p < q < r$ such that the function f satisfies the following conditions:*
(i) $f(t, (1+t)u, v) > \varphi(rN)$ for $(t, u, v) \in [a, \infty)_{\mathbb{T}} \times [0, Mr] \times [0, r]$,
(ii) $f(t, (1+t)u, v) < \varphi(\frac{qN}{M})$ for $(t, u, v) \in [a, \infty)_{\mathbb{T}} \times [0, q] \times [0, q]$,
(iii) $f(t, (1+t)u, v) > \varphi(pN)$ for $(t, u, v) \in [a, \infty)_{\mathbb{T}} \times [0, p] \times [0, p]$,
where N and M are defined in (9) and (7), respectively. Then the BVP (1) has at least two positive solutions y_1 and y_2 such that

$$\|y_1\| > p \text{ with } \|y_1\| < q \text{ and } \|y_2\| > q \text{ with } y_2^{\Delta}(a) < r.$$

Proof. Define the cone P as in (6). From Lemma 2.3, $A : P \to P$ is completely continuous. Let the nonnegative increasing continuous functionals ϕ, θ and η be defined on the cone P by

$$\phi(y) := y^{\Delta}(a), \;\; \theta(y) := \|y\|, \;\; \eta(y) := \|y\|.$$

For each $y \in P$, we have

$$\phi(y) \leq \theta(y) = \eta(y)$$

and from (8) we have

$$\|y\| \leq M\|y^{\Delta}\|_{\infty} = My^{\Delta}(a) = M\phi(y).$$

In addition, $\theta(0) = 0$ and for all $y \in P$, $\lambda \in [0,1]$ we get $\theta(\lambda y) = \lambda\theta(y)$. We now verify that all of the conditions of Theorem 3.3 are satisfied.

If $y \in \partial P(\phi, r)$, for $s \in [a,\infty)_{\mathbb{T}}$ we have $0 \leq y^{\Delta}(s) \leq r$ and $0 \leq \frac{y(s)}{1+s} \leq Mr$ from Lemma 2.2. Then, from the hypothesis (*i*) and (9), we find

$$\begin{aligned} \phi(Ay) &= \varphi^{-1}\left(\int_a^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right) \nabla s\right) \\ &> rN\varphi^{-1}\left(\int_a^{\infty} h(s)\nabla s\right) \\ &= r. \end{aligned}$$

Thus the condition (*i*) of Theorem 3.3 holds.

If $y \in \partial P(\theta, q)$, we have $0 \leq \frac{y(s)}{1+s} \leq q$ and $0 \leq y^{\Delta}(s) \leq q$ for $s \in [a,\infty)_{\mathbb{T}}$. Then, we obtain

$$\begin{aligned} \theta(Ay) &\leq M\varphi^{-1}\left(\int_a^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right) \nabla s\right) \\ &< q \end{aligned}$$

by hypothesis (*ii*), (8) and (9). Hence the condition (*ii*) of Theorem 3.3 is satisfied.

Since $0 \in P$ and $p > 0$, $P(\eta, p) \neq \emptyset$. If $y \in \partial P(\eta, p)$, we have $0 \leq \frac{y(s)}{1+s} \leq p$ and $0 \leq y^{\Delta}(s) \leq p$ for $s \in [a,\infty)_{\mathbb{T}}$. Then, we get

$$\begin{aligned} \eta(Ay) &\geq \left\|(Ay)^{\Delta}\right\|_{\infty} \\ &= \varphi^{-1}\left(\int_a^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right) \nabla s\right) \\ &> p \end{aligned}$$

using hypothesis (*iii*) and (9). Since all the conditions of Theorem 3.3 are fulfilled, the BVP (1) has at least two positive solutions y_1 and y_2 such that

$$\|y_1\| > p \text{ with } \|y_1\| < q \;\text{ and }\; \|y_2\| > q \text{ with } y_2^{\Delta}(a) < r.$$

□

Now, we will present the five functionals fixed point theorem. Let γ, ϕ, θ be nonnegative continuous convex functionals on the cone P, and α, Ψ nonnegative continuous concave functionals on the cone P. For nonnegative numbers b, d, m, l and c, define the following convex sets:

$$\begin{cases} P(\gamma, c) = \{y \in P : \gamma(y) < c\}, \\ P(\gamma, \alpha, m, c) = \{y \in P : m \leq \alpha(y), \gamma(y) \leq c\}, \\ Q(\gamma, \phi, d, c) = \{y \in P : \phi(y) \leq d, \gamma(y) \leq c\}, \\ P(\gamma, \theta, \alpha, m, b, c) = \{y \in P : m \leq \alpha(y), \theta(y) \leq b, \gamma(y) \leq c\}, \\ Q(\gamma, \phi, \Psi, l, d, c) = \{y \in P : l \leq \Psi(y), \phi(y) \leq d, \gamma(y) \leq c\}. \end{cases} \tag{10}$$

The following theorem can be found in [2].

Theorem 3.5. *(Five Functionals Fixed Point Theorem) Let P be a cone in a real Banach space E. Suppose that there exist nonnegative numbers c and r, nonnegative continuous concave functionals α and Ψ on P, and nonnegative continuous convex functionals γ, ϕ and θ on P, with*

$$\alpha(y) \leq \phi(y), \|y\| \leq r\gamma(y), \forall y \in \overline{P(\gamma, c)}.$$

Suppose that $A : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ is a completely continuous and there exist nonnegative numbers b, d, m, l with $0 < d < m$ such that

(i) $\{y \in P(\gamma, \theta, \alpha, m, b, c) : \alpha(y) > m\} \neq \emptyset$ *and* $\alpha(Ay) > m$ *for* $y \in P(\gamma, \theta, \alpha, m, b, c)$,
(ii) $\{y \in Q(\gamma, \phi, \Psi, l, d, c) : \phi(y) < d\} \neq \emptyset$ *and* $\phi(Ay) < d$ *for* $y \in Q(\gamma, \phi, \Psi, l, d, c)$,
(iii) $\alpha(Ay) > b$, *for* $y \in P(\gamma, \alpha, m, c)$, *with* $\theta(Ay) > b$,
(iv) $\phi(Ay) < d$, *for* $y \in Q(\gamma, \phi, d, c)$, *with* $\Psi(Ay) < l$,

then A has at least three fixed points $y_1, y_2, y_3 \in \overline{P(\gamma, c)}$ such that

$$\phi(y_1) < d, \alpha(y_2) > m, \phi(y_3) > d \text{ with } \alpha(y_3) < m.$$

Define the constant

$$\lambda = \varphi^{-1}\left(\int_{\frac{1}{k}}^{k} h(s)\nabla s\right). \tag{11}$$

Now, we will apply the five functionals fixed point theorem to investigate the existence of at least three positive solutions to the nonlinear BVP (1).

Theorem 3.6. *Assume (H1)-(H3) hold and $\frac{1}{k} \in \mathbb{T}$ for each $k \in \{1, 2, \cdots n\}$. Suppose that there exist constants $0 < d < m < c$ such that the function f satisfies the following conditions:*
(i) $f(t, (1+t)u, v) > \varphi\left(\frac{(k+1)m}{(1-ak)\lambda}\right)$ *for* $(t, u, v) \in [\frac{1}{k}, k]_{\mathbb{T}} \times [\frac{m}{k}, c] \times [0, c]$,
(ii) $f(t, (1+t)u, v) < \varphi(\frac{dN}{M})$ *for* $(t, u, v) \in [a, \infty)_{\mathbb{T}} \times [0, d] \times [0, d]$,
(iii) $f(t, (1+t)u, v) \leq \varphi(\frac{cN}{M})$ *for* $(t, u, v) \in [a, \infty)_{\mathbb{T}} \times [0, c] \times [0, c]$,
Then the BVP (1) has at least three positive solutions y_1, y_2 and y_3 satisfying

$$\|y_1\| < d < \|y_3\|, \text{ and } \frac{k}{k+1} \min_{t \in [\frac{1}{k}, \infty)_{\mathbb{T}}} y_3(t) < m < \frac{k}{k+1} \min_{t \in [\frac{1}{k}, \infty)_{\mathbb{T}}} y_2(t).$$

Proof. Define the cone P as in (6). Let $l = 0$, $r = 1$ and define the nonnegative, continuous, concave functionals α, ψ and the nonnegative, continuous, convex functionals γ, ϕ, θ on P by

$$\alpha(y) = \frac{k}{k+1} \min_{t \in [\frac{1}{k}, \infty)_{\mathbb{T}}} y(t), \quad \gamma(y) = \phi(y) = \theta(y) = \|y\|, \quad \psi(y) = 0.$$

Let $P(\gamma, c)$, $P(\gamma, \alpha, m, c)$, $Q(\gamma, \phi, d, c)$, $P(\gamma, \theta, \alpha, m, b, c)$ and $Q(\gamma, \phi, \Psi, l, d, c)$ be defined by (10). It is clear that

$$\alpha(y) < \phi(y), \ \ \|y\| = \gamma(y), \ \ \forall y \in \overline{P(\gamma, c)}.$$

If $y \in \overline{P(\gamma, c)}$, then we have $0 \le \frac{y(t)}{1+t} \le c$ and $0 \le y^{\Delta}(t) \le c$ for all $t \in [a, \infty)_{\mathbb{T}}$. By (8), (9) and the hypothesis (iii), we get

$$\begin{aligned} \gamma(Ay) &\le M\|(Ay)^{\Delta}\|_{\infty} \\ &\le M\varphi^{-1}\left(\int_a^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\right) \\ &\le c. \end{aligned}$$

This proves that $A : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$. From Lemma 2.3, $A : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ is completely continuous.

Now we verify that the remaining conditions of Theorem 3.5.

If we take $y(t) = \frac{c+m}{2}(t+1)$ for $t \in [a, \infty)_{\mathbb{T}}$, we obtain $y \in P$, $\alpha(y) = \frac{c+m}{2} > m$ and $\|y\| = \frac{c+m}{2} < c = b$. That is, $\{y \in P(\gamma, \theta, \alpha, m, b, c) : \alpha(y) > m\} \neq \emptyset$.

If $y \in P(\gamma, \theta, \alpha, m, b, c)$, then for all $t \in [\frac{1}{k}, k]_{\mathbb{T}}$ we have $\frac{m}{k} \le \frac{y(t)}{1+t} \le c$ and $0 \le y^{\Delta}(t) \le c$. By using (11) and the hypothesis (i), we get

$$\begin{aligned} \alpha(Ay) &= \frac{k}{k+1}\Bigg[\sum_{i=1}^{m-2}\alpha_i\varphi^{-1}\left(\int_{\eta_i}^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\right) + \beta\varphi^{-1}\left(\int_a^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\right) \\ &+ \int_a^{\frac{1}{k}}\varphi^{-1}\left(\int_{\xi}^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\right)\Delta\xi + \sum_{a<t_k<\frac{1}{k}} I_k(y(t_k))\Bigg] \\ &\ge \frac{k}{k+1}\int_a^{\frac{1}{k}}\varphi^{-1}\left(\int_{\frac{1}{k}}^{k} h(s) f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\right)\Delta\xi \\ &= \frac{1-ak}{k+1}\varphi^{-1}\left(\int_{\frac{1}{k}}^{k} h(s) f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\right) \\ &> m. \end{aligned}$$

Then, we have

$$\alpha(Ay) > m. \tag{12}$$

Thus, the condition (i) of Theorem 3.5 holds.

If we take $y(t) = \frac{d}{2}$ for $t \in [a, \infty)_{\mathbb{T}}$, we obtain $y \in P$, $l = 0 = \psi(y)$ and $\|y\| = \frac{d}{2} < d < c$. That is, $\{y \in Q(\gamma, \phi, \Psi, l, d, c) : \phi(y) < d\} \neq \emptyset$.

If $y \in Q(\gamma, \phi, \psi, l, d, c)$, then for all $t \in [a, \infty)_{\mathbb{T}}$ we obtain $0 \le \frac{y(t)}{1+t} \le d$ and $0 \le y^{\Delta}(t) \le d$. Hence,

$$\begin{aligned} \phi(Ay) &\le M\|(Ay)^{\Delta}\|_{\infty} \\ &\le M\varphi^{-1}\left(\int_a^{\infty} h(s) f\left(s, y(s), y^{\Delta}(s)\right)\nabla s\right) \\ &< d \end{aligned}$$

by (8), (9) and the hypothesis (*ii*). It follows that the condition (*ii*) of Theorem 3.5 is fulfilled.

Now, we shall show that the condition (*iii*) of Theorem 3.5 is satisfied. If $y \in P(\gamma, \alpha, m, c)$, then for all $t \in [\frac{1}{k}, k]_{\mathbb{T}}$ we have $\frac{m}{k} \leq \frac{y(t)}{1+t} \leq c$ and $0 \leq y^{\Delta}(t) \leq c$. According to (12), we have $\alpha(Ay) > m$. Thus, the condition (*iii*) of Theorem 3.5 holds.

Finally, we shall verify that the condition (*iv*) of Theorem 3.5 holds. Since $\psi(Ay) < l = 0$ is impossible, we omit the condition (*iv*) of Theorem 3.5.

Since all the conditions of Theorem 3.5 are satisfied, the BVP (1) has at least three positive solutions y_1, y_2 and y_3 satisfying

$$\|y_1\| < d < \|y_3\|, \text{ and } \frac{k}{k+1} \min_{t \in [\frac{1}{k}, \infty)_{\mathbb{T}}} y_3(t) < m < \frac{k}{k+1} \min_{t \in [\frac{1}{k}, \infty)_{\mathbb{T}}} y_2(t).$$

This completes the proof.

□

Example 3.7. *Let $\mathbb{T} = [0,3] \cup [8, \infty)$. Consider the following boundary value problem:*

$$\begin{cases} y^{\Delta\nabla}(t) + e^{-t} \frac{100t^2}{1+t^2} \left(\frac{y^2(t)}{(1+t)^2} \left(y^{\Delta}(t) \right)^2 \right) = 0, \quad t \neq 2, \quad t \in [0, \infty) \subset \mathbb{T} \\ y(2^+) - y(2^-) = 4, \\ y(0) - 2y^{\Delta}(0) = \frac{1}{2} y^{\Delta}\left(\frac{1}{2}\right) + y^{\Delta}(1), \quad \lim_{t \to \infty} y^{\Delta}(t) = 0. \end{cases}$$

Taking $\varphi(x) = x$, $h(x) = e^{-x}$, $a = 0$, $t_1 = \beta = k = 2$, $\alpha_1 = \eta_1 = \frac{1}{2}$ and $\alpha_2 = \eta_2 = 1$, we have $M = \frac{7}{2}$, $N = \frac{e^8}{e^8 - e^5 + 6}$ and $\lambda = \frac{e^{3/2}-1}{e^2}$. If we take $d = 0.001, m = 0.0013$ and $c = 0.0015$, then all the conditions in Theorem 3.6 are verified. Thus, the BVP has at least three positive solutions y_1, y_2 and y_3 satisfying

$$\|y_1\| < d < \|y_3\|, \text{ and } \frac{2}{3} \min_{t \in [\frac{1}{2}, \infty)_{\mathbb{T}}} y_3(t) < m < \frac{2}{3} \min_{t \in [\frac{1}{2}, \infty)_{\mathbb{T}}} y_2(t).$$

Acknowledgment The authors thank the referees for their valuable suggestions and comments.

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