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VIBRATION ANALYSIS OF FUNCTIONALLY GRADED SANDWICH BEAM WITH VARIABLE CROSS-SECTION

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Abstract- In this study, free vibration behavior of a multilayered symmetric sandwich beam made of Functionally Graded Material (FGM) with variable cross-section is investigated. The elasticity and density of the Functionally Graded (FG) sandwich beam vary through the thickness according to the power and exponential laws by using mixture rules and laminate theory. In order to provide this, fifty layered beam is considered. Each layer is isotropic and homogeneous although the volume fractions of the constituents of the layers are different. Furthermore, the width of the beam varies exponentially along the length of the beam with rectangular cross-section. The natural frequencies are computed for conventional boundary conditions of the FG sandwich beam using theoretical procedure. The effects of material index, geometric index and slenderness ratio are also discussed. Finally, the obtained results are compared with those in literature and a finite element based commercial program ANSYS® and found to be consistent with each other.

Key Words- Functionally Graded Materials, Free Vibration, Variable Cross-Section, Sandwich Beam

1. INTRODUCTION

The Functionally Graded Materials (FGMs) are obtained by changing the volume fractions of constituents from one surface to the other gradually. So, the material properties of the structure can be adjusted according to demand. Due to this advantage, these materials have attracted the attention of many researchers. The field has been developed rapidly due to their wide practical application in machine, civil, aerospace and automotive areas.

Some studies have been performed to analyze the behaviors of FGM structures, e.g. [1-4]. Free vibrations of beams made of FGMs have been studied by some researchers. Aydogdu and Taskin [5] investigated free vibration of simply supported FG beam by using parabolic, first order and exponential shear deformation beam theories. Aydogdu [6] used Semi-inverse Method to find a relation between elasticity modulus and natural frequency and buckling. Pradhan and Murmu [7] presented thermo-mechanical vibration analysis of beams and sandwich beams made of FGM under different conditions. Bedjilili *et al*. [8] coped with the free vibration of composite beams with a variable fiber volume fraction using the first-order shear deformation theory. A forth order differential equation of a homogenized beam deflection was dealt with by Murin *et al.* [9]. Mahi *et al.* [10] analyzed free vibration of symmetric FGM beam subjected to initial thermal stresses by using a theoretical formulation and they assumed the material properties as temperature-dependent. Additionally, there are some studies related to free vibration behaviors of beams with variable cross section. Ece *et al.* [11] investigated the vibration of an isotropic beam which has a variable cross-section. Atmane *et al*. [12] presented a theoretical investigation for free vibration of a functionally graded beam with variable cross-section. Their theory is based on Kirchhoff-Love hypothesis and they only changed the material properties exponentially.

Cranch and Adler [13] presented the closed-form solutions for the natural frequencies and mode shapes of the unconstrained non-uniform beams with four kinds of rectangular cross-sections. Caruntu [14] examined the nonlinear vibrations of beams with rectangular cross-section and parabolic thickness variation. Datta and Sil [15] numerically determined the natural frequencies of cantilever beams with constant width and linearly varying depth. Laura *et al.* [16] used approximate numerical approaches to determine the natural frequencies of Bernoulli beams with constant width and bilinearly varying thickness.

In this study, free vibration behavior of a symmetric FG sandwich beam with variable cross-section is analyzed. The material properties of the FG sandwich beam vary through the thickness according to the power and exponential laws. As a result of this, effective elasticity modulus and mass density are obtained by using mixture rules and laminate theory. The width of the beam with rectangular cross-section changes exponentially along the length of the FG beam. Natural frequencies are found by using obtained effective material properties for various boundary conditions. The results obtained are compared with both ANSYS® solutions and studies in the literature. All results obtained are found to be consistent with each other. The effects of material and geometrical indexes and slenderness ratio on the vibration behaviors of the sandwich beam with variable cross-section are also discussed.

2. DETERMINATION OF THE EFFECTIVE MATERIAL AND GEOMETRY PROPERTIES

Consider a transversely vibrating symmetric sandwich beam with variable cross-section made of functionally graded material as shown in Figure 1.

Here; L and h represent length and thickness of the beam, and b_0 is half width at the left end of the beam. However, the width of the beam (b(x)) is supposed to vary exponentially along the length of the beam as follows,

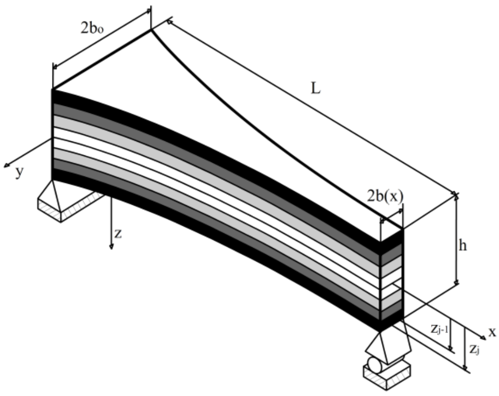


Figure 1. A FG sandwich beam with variable cross-section

$$b(x) = b_0 e^{\beta x} \tag{1}$$

where β is geometric index and as β = 0, the beam has uniform cross-section.

The beam is assumed to compose of fifty FG layers in order to get more consistent value in the solution. Each layer is a mixture of Aluminum (Al) and Alumina (Al_2O_3) phases and layers are arranged symmetrically to the neutral plane of the beam. That is, the beam is stacked as $[Al_2O_3/FGM/Al]_s$. The mixture ratio is chosen as a polynomial or an exponential function, and it is varied continuously and symmetrically through the thickness with respect to the neutral plane of the beam.

In order to obtain the effective material properties of the whole structure, following procedure is applied. Firstly, the material properties of the upper half of the beam are calculated from exponential and power laws as given in Eqs. (2) and (3). Secondly, effective material properties of the whole structure are obtained by using the formula of effective elasticity modulus for symmetric laminated composite structures. The exponential and power laws for elasticity modulus are given, respectively, as follow,

$$E(z) = E_c e^{(-\delta(1-2z))}, \quad \delta = \frac{1}{2}\ln\left(\frac{E_c}{E_m}\right) \tag{2}$$

$$E(z) = (E_c - E_m)\left(z + \frac{1}{2}\right)^n + E_m \tag{3}$$

where E_c, E_m, z and n are elasticity moduli of ceramic and metal phases, the coordinate axis in the thickness direction of the beam and material index, respectively. The variation of mass density in each layer through the beam thickness has also been considered to obtain more accurate results. The expressions written for elasticity moduli are also considered to be valid for density.

$$\rho(z) = \rho_c e^{(-\delta(1-2z))}, \quad \delta = \frac{1}{2}\ln\left(\frac{\rho_c}{\rho_m}\right) \tag{4}$$

$$\rho(z) = (\rho_c - \rho_m)\left(z + \frac{1}{2}\right)^n + \rho_m \tag{5}$$

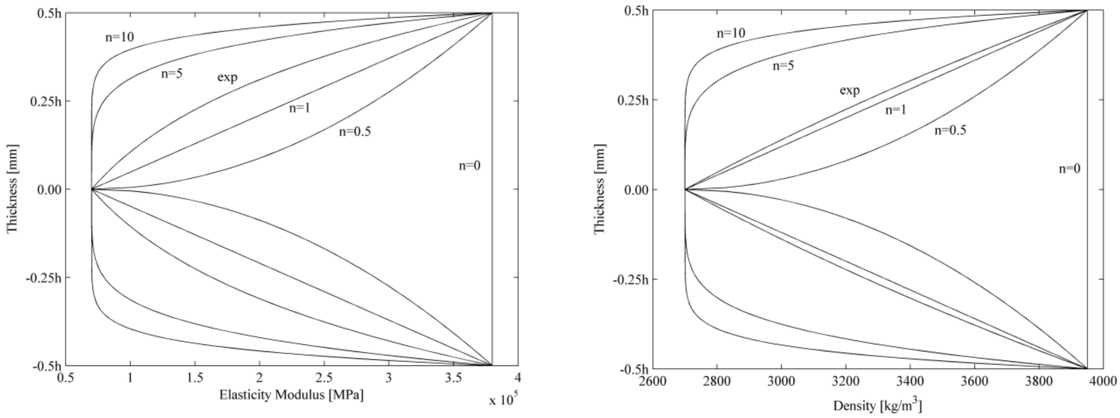
The above-mentioned variable z is defined as z = -1/2,-1/2+1/η,-1/2+2/η,…,1/2, where η is equal to (m/2)-1, where m represents the number of the layer of the beam.

It is seen from Figure 1 that top and bottom surfaces of FG beam are pure ceramic, whereas the middle section of the beam is pure metal. The variations of elasticity modulus and the mass density through the whole thickness of the beam for various material indexes (n) are shown in Figure 2.

Eqs. (2-5) give elasticity modulus and mass density for each layer in upper half-part of the beam. In order to find the elasticity modulus and the mass density throughout the whole beam thickness, the classical laminate theory will be used.

The bending moment on a symmetric FG sandwich beam can be written in a similar way used in laminated composite beam theory [17],

$$M = \frac{2b}{3r}\sum_{j=1}^{m/2}\left(E_z\right)_j\left(z_j^3 - z_{j-1}^3\right) \tag{6}$$



(a) (b)

Figure 2. Variations of elasticity modulus (a) and mass density (b) through the thickness for exponential-law and different index values of power-law functions

where r and z_j are curvature of the beam and distance between the outer face of jth layer and the neutral plane, respectively. The bending moment can also be written as follows,

$$M = \frac{E_{ef} I_{yy}}{r} \tag{7}$$

where E_{ef} is the effective elasticity modulus and I_{yy} is the cross-sectional inertia moment about the neutral axis of the beam. Substituting Eq. (6) into Eq. (7), the effective elasticity modulus can be written as,

$$E_{ef} = \frac{8}{h^3}\sum_{j=1}^{m/2}\left(E_z\right)_j\left(z_j^3 - z_{j-1}^3\right) \tag{8}$$

Similarly, effective mass density can be written as follows,

$$\rho_{ef} = \frac{8}{h^3}\sum_{j=1}^{m/2}\left(\rho_z\right)_j\left(z_j^3 - z_{j-1}^3\right) \tag{9}$$

In the calculation of the natural frequency, the effective elasticity modulus E_{ef} and the effective mass density ρ_{ef} can be used instead of elasticity modulus E and mass density ρ in a beam manufactured from isotropic and homogenous materials.

3. THEORETICAL FORMULATION AND SOLUTION

The bending moment can be written from equilibrium equations of the beam as,

$$\frac{\partial^2 M}{\partial x^2} = \rho_{ef} A(x) \frac{\partial^2 w(x,t)}{\partial t^2} \tag{10}$$

and,

$$M = -E_{ef} I(x) \frac{\partial^2 w(x,t)}{\partial x^2} \tag{11}$$

where A(x) = b(x) h is variable cross-sectional area, I(x) = b(x) h^3 / 12 is variable inertia moment and t is time. The transverse displacement w(x,t) is expressed as,

$$w(x,t) = W(x) e^{i\omega t} \tag{12}$$

Substituting Eqs.(11) and (12) into Eq.(10) yields,

$$\frac{d^4 W(x)}{dx^4} + 2\beta \frac{d^3 W(x)}{dx^3} + \beta^2 \frac{d^2 W(x)}{dx^2} - \alpha^2 W(x) = 0 \tag{13}$$

where $\alpha^2 = 12\ \rho_{ef}\ \omega^2 / E_{ef}\ h^2$.

As Eq.(13) is solved, the following expression can be obtained as,

$$W(x) = e^{-\frac{\beta}{2}x} \left(C_1 e^{i\lambda_1 x} + C_2 e^{-i\lambda_1 x} + C_3 e^{\lambda_2 x} + C_4 e^{-\lambda_2 x} \right) \tag{14}$$

here, $\lambda_1 = \sqrt{4\alpha - \beta^2} / 2$ and $\lambda_2 = \sqrt{4\alpha + \beta^2} / 2$ and C_1, C_2, C_3 and C_4 are (complex) constants. In other words, the solution can also be written as trigonometric form,

$$W(x) = e^{-\frac{\beta}{2}x} \left(B_1 \cos(\lambda_1 x) + B_2 \sin(\lambda_1 x) + B_3 \cosh(\lambda_2 x) + B_4 \sinh(\lambda_2 x) \right) \tag{15}$$

where B_1, B_2, B_3 and B_4 are real constants and they are determined by the boundary conditions.

Three different types of boundary conditions are considered in this study: clamped (C), simply supported (S) and free (F). These boundary condition types are described as,

$$C \quad \rightarrow \quad W = 0; \quad dW / dX = 0 \tag{16}$$

$$S \quad \rightarrow \quad W = 0; \quad d^2W / dX^2 = 0 \tag{17}$$

$$F \quad \rightarrow \quad d^2W / dX^2 = 0; \quad d^3W / dX^3 = 0 \tag{18}$$

The widely used boundary conditions are taken into account at two ends of the beam, i.e., C-C, C-S, S-S and C-F.

As the boundary conditions are applied to Eq.(15), four equations are emerged for each boundary conditions. The roots of these four equations are obtained by its determinant. As a result of this, the characteristic equations can be found for each boundary condition. For example, the characteristic equation of C-C boundary condition is as follow,

$$\begin{aligned} &-e^{-\beta L} \left(-2\lambda_1 \lambda_2 - \sin(\lambda_1 L) \lambda_2^2 \sinh(\lambda_2 L) + 2\cos(\lambda_1 L) \lambda_1 \cosh(\lambda_2 L) \lambda_2 \right. \\ &\left. + \sin(\lambda_1 L) \lambda_1^2 \sinh(\lambda_2 L) \right) = 0 \end{aligned} \tag{19}$$

In order to calculate the natural frequencies, these characteristic equations are solved numerically.

4. RESULTS AND DISCUSSION

In this study, the fifty layered FG sandwich beam with variable cross-section is considered. The material properties of the constituents of the beam are given in Table 1.

The elasticity modulus and density of FG beam are taken to be variable and Poisson's ratio υ can be considered as a constant. Additionally, the width of the FG beam is also assumed to be exponentially variable along the length of the beam. The thickness and the length of the beam are constant and they are set to h = 5 mm and L = 200 mm, respectively. The width at the left end of the FG beam is b_0 = 20 mm. Material index (n) is considered as 0, 0.5, 1, 5 and 10, and geometric index (β) is -1/L, -0.5/L, 0.0, 0.5/L and 1/L. The FG beam is isotropic, homogenous and uniform beam as n and β are equal to zero.

Table 1. Material properties of the constituents of the FG sandwich beam

Material	E (GPa)	ρ (kg/m3)	υ
Al	70	2700	0.3
Al2O3	380	3950	0.3

The dimensionless natural frequencies for isotropic, homogenous beam of β =-1/L under C-F boundary conditions is compared with ones in literature and given in Table 2. It is seen that they are in conformity with each other.

Table 2. Dimensionless natural frequencies for isotropic homogenous beam of β = -1/L under C-F boundary conditions

Mode	Present	Ref. [11]	Ref. [12]	Ref. [13]	Ref. [18]
1	4.73491	4.72298	4.72300	4.73500	4.73470
2	24.20181	24.20168	24.20170	24.20250	24.20050
3	63.86449	63.86448	63.86450	63.85000	63.86080
4	123.09791	123.09790	123.09800	-	123.09100

In order to support the accuracy of the results obtained from present method, the FG sandwich beam is also solved by the commercial program ANSYS® using Finite element analyses. The beam is modeled in SOLIDWORKS® and imported to ANSYS®. The model is meshed by SOLID186 (3-D 20-Node structural solid) elements. SOLID186 is well suited to modeling irregular meshes. The material properties (elasticity modulus and density) of this element are taken from analytical calculations as effective values. The Block Lanczos method is used for the eigenvalue extractions.

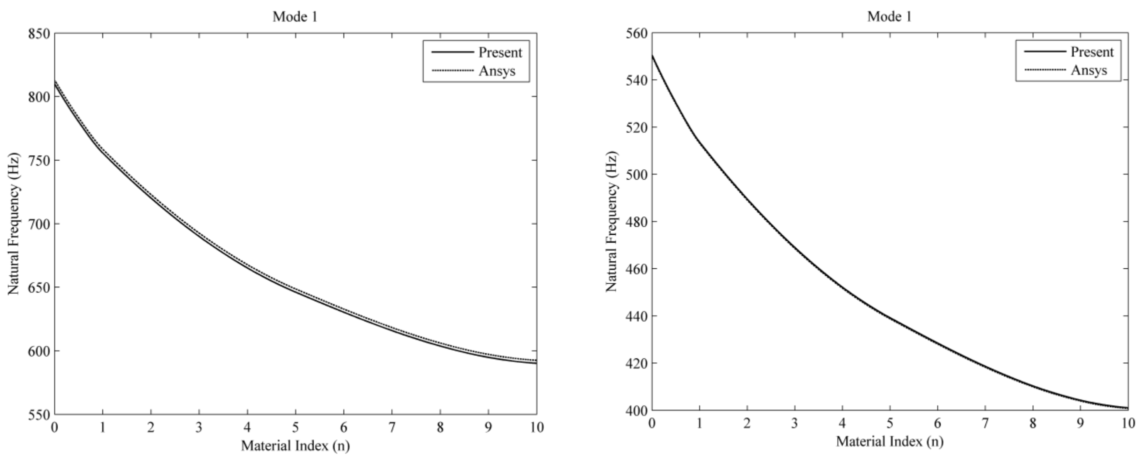
4.1. Effect of material index

Figure 3 illustrates the variations of the natural frequencies of FG beam versus material index (n) for C-S and S-S boundary conditions and geometric index β = 1/L.

It can be seen that natural frequencies decrease with increasing material index (n) for both boundary conditions. Namely, increase in the volume fractions of the ceramic phase in the FG sandwich beam causes increase in the natural frequencies. As a result of this, in order to achieve desired natural frequencies, the volume fractions of the constituents of the symmetric FG sandwich beam can be arranged. In order to verify the accuracy of the results obtained from present method, corresponding beam is also solved by ANSYS®. The results obtained from both methods are very close to each other.

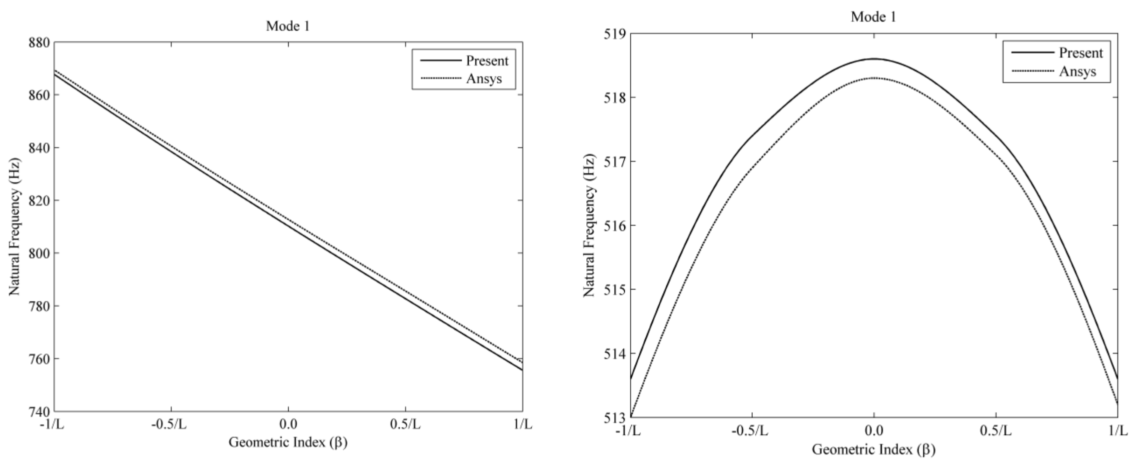
4.2. Effect of geometric index

Figure 4 shows the variations of the natural frequencies of FG beam with geometric index (β) for C-S and S-S boundary conditions and material index n = 1. It can be seen from Figure 4(a) that natural frequencies gradually decrease with changing from narrowing to expanding of the cross-section for C-S beam. As for S-S beam, the natural frequencies of the beam with the narrowing or expanding cross-sections are symmetrical according to the one with uniform cross-section, as shown in Figure 4(b). As a result, for symmetrical boundary conditions, narrowing beams can provide more advantages than expanding beams because less material is used. It is also seen that the results of present and ANSYS® solutions are very close to each other.



(a) (b)

Figure 3. Variations of the natural frequencies with material index (n) for C-S (a) and S-S (b) boundary conditions and β = 1/L



(a) (b)

Figure 4. Variations of the natural frequencies with geometric index (β) for C-S (a) and S-S (b) boundary conditions and n = 1

4.3. Effect of the combining of both indexes

The first three natural frequencies of the symmetric FG sandwich beam under different three type boundary conditions are listed in Table 3 for geometric index, material index and its function types. With increasing n and β, the natural frequencies gradually decrease in C-F beam for all mode numbers. As for the beam with symmetrical boundary conditions i.e. S-S and C-C, their natural frequencies are symmetrical according to the beam of uniform cross-section by increasing n and β, as also seen in Figure 4(b).

Table 3. First three frequencies (Hz) of the symmetric FG sandwich beam

β	Type	n	C-F			S-S			C-C		
			ω_1	ω_2	ω_3	ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
-1/L	Exp.	exp	233.6	1194.	3150.	482.2	1952.	4389.	1110.	3052.	5975.
	Polynomial	0	266.7	1363.	3597.	550.5	2229.	5011.	1268.	3484.	6821.
		0.5	256.9	1313.	3465.	530.3	2147.	4828.	1221.	3356.	6572.
		1	248.8	1271.	3356.	513.6	2079.	4675.	1183.	3250.	6364.
		5	212.8	1087.	2869.	439.2	1778.	3998.	1011.	2779.	5442.
		10	194.3	993.4	2621.	401.1	1624.	3651.	924.0	2539.	4971.
-0.5/L	Exp.	exp	201.8	1139.	3095.	485.7	1948.	4384.	1105.	3045.	5967.
	Polynomial	0	230.3	1300.	3534.	554.6	2225.	5005.	1262.	3476.	6813.
		0.5	221.9	1253.	3404.	534.3	2143.	4822.	1216.	3349.	6563.
		1	214.9	1213.	3296.	517.4	2075.	4669.	1177.	3243.	6356.
		5	183.8	1037.	2819.	442.4	1775.	3993.	1006.	2773.	5435.
		10	167.8	947.9	2575.	404.1	1621.	3647.	919.7	2533.	4964.
0	Exp.	exp	173.5	1087.	3044.	486.9	1947.	4382.	1103.	3042.	5965.
	Polynomial	0	198.1	1241.	3475.	555.9	2223.	5003.	1260.	3474.	6810.
		0.5	190.8	1195.	3348.	535.6	2142.	4820.	1214.	3346.	6560.
		1	184.8	1157.	3242.	518.6	2074.	4667.	1175.	3240.	6353.
		5	158.0	990.2	2772.	443.5	1774.	3991.	1005.	2771.	5433.
		10	144.3	904.4	2532.	405.1	1620.	3646.	918.3	2531.	4962.
0.5/L	Exp.	exp	148.5	1037.	2996.	485.7	1948.	4384.	1105.	3045.	5967.
	Polynomial	0	169.5	1183.	3421.	554.6	2225.	5005.	1262.	3476.	6813.
		0.5	163.3	1140.	3296.	534.3	2143.	4822.	1216.	3349.	6563.
		1	158.2	1104.	3191.	517.4	2075.	4669.	1177.	3243.	6356.
		5	135.3	944.5	2729.	442.4	1775.	3993.	1006.	2773.	5435.
		10	123.5	862.7	2493.	442.4	1621.	3647.	919.7	2533.	4964.
1/L	Exp.	exp	126.6	988.7	2953.	482.2	1952.	4389.	1110.	3052.	5975.
	Polynomial	0	144.5	1128.	3372.	550.5	2229.	5011.	1268.	3484.	6821.
		0.5	139.2	1087.	3248.	530.3	2147.	4828.	1221.	3356.	6572.
		1	134.8	1053.	3146.	513.6	2079.	4675.	1183.	3250.	6364.
		5	115.3	900.5	2690.	439.2	1778.	3998.	1011.	2779.	5442.
		10	105.3	822.5	2457.	401.1	1624.	3651.	924.0	2539.	4971.

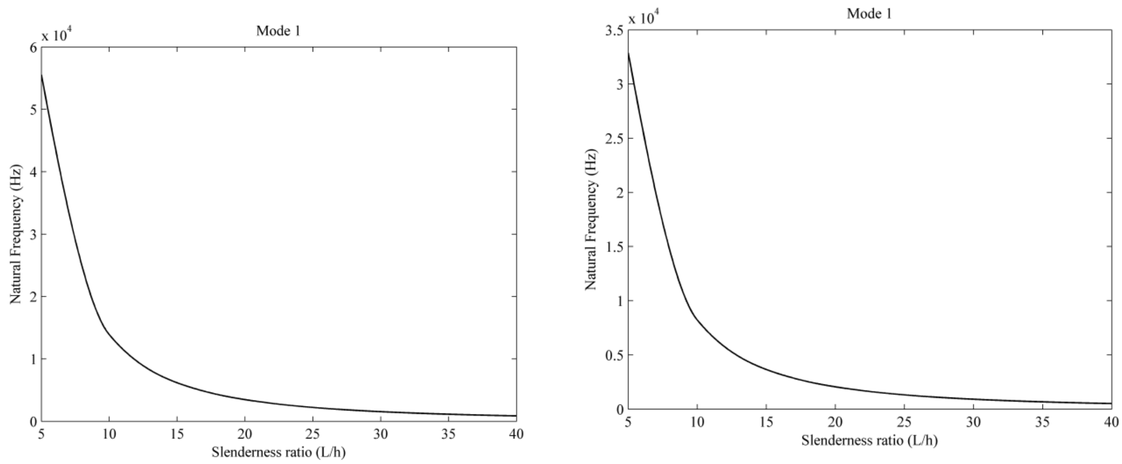
4.4. Effect of the slenderness ratio

The variations of the natural frequencies against the slenderness ratio (L/h) for material index n = 1 and geometric index β = -1/L are depicted in Figure 5. As expected, natural frequencies decrease with increasing slenderness ratio for both C-S and S-S boundary conditions. Especially, while the natural frequencies of the beam reduce sharply for slenderness ratio between 5 and 10, they decline slowly after L/h = 10.

5. CONCLUSIONS

The free vibration of a symmetric FG sandwich beam with variable cross-section is investigated in this paper, and the verification is carried out by comparing the results in literature and obtained from ANSYS® commercial software. The following conclusions can be drawn from the analyses:

- The effective elasticity modulus and the effective mass density can be used instead of elasticity modulus and mass density in the governing equations, which belong to the natural frequencies, of the conventional materials.
- Increase in the volume fractions of the ceramic in the symmetric FG sandwich beam causes increase in the natural frequencies.
- The natural frequencies decrease gradually with increasing material index.
- The natural frequencies vary symmetrically according to one of uniform cross-section for symmetric boundary conditions i.e. C-C and S-S with increasing geometric index. Whereas, the natural frequencies for other boundary conditions decreases gradually.
- The natural frequencies decrease drastically with increasing slenderness ratio for all boundary conditions.
- It is found that present results are agreed with the results of the other methods in literature and ANSYS®.



(a) (b)

Figure 5. Variations of the natural frequencies with slenderness ratio (L/h) for C-S (a) and S-S (b) boundary conditions, n = 1 and β = -1/L

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