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BERNSTEIN COLLOCATION METHOD FOR SOLVING NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract- In this study, a collocation method based on Bernstein polynomials is developed for solution of the nonlinear ordinary differential equations with variable coefficients, under the mixed conditions. These equations are expressed as linear ordinary differential equations via quasilinearization method iteratively. By using the Bernstein collocation method, solutions of these linear equations are approximated. Combining the quasilinearization and the Bernstein collocation methods, the approximation solution of nonlinear differential equations is obtained. Moreover, some numerical solutions are given to illustrate the accuracy and implementation of the method.

Key Words- Bernstein polynomial approximation, Quasilinearization technique, Nonlinear differential equations, Collocation method

1. INTRODUCTION

The quasilinearization method [2, 4, 12] based on the Newton-Raphson method is an effective approximation technique for solution of the nonlinear differential equations and partial differential equations. Aim of this method is to solve a nonlinear *n*th order ordinary or partial differential equation in *N* dimensions as a limit of a sequence of linear differential equations. So it is a powerful tool that nonlinear differential equations are expressed as a sequence of linear differential equations. This method also provides a sequence of functions which converges rather rapidly to the solutions of the original nonlinear equations. Moreover, this method has been applied to a variety of problems involving different equations like nonlinear initial and boundary value problems involving functional differential equations [1], functional differential equations with retardation and anticipation [5], singular boundary value problems [11], nonlinear Volterra integral equations [8,10], mix integral equations [3], integro-differential equation [13].

The Bernstein polynomials and their basis form that can be generalized on the interval $[a,b]$, are defined as follows:

Definition 1.1 Generalized Bernstein basis polynomials can be defined on the interval $[a, b]$; by

$$p_{i,n}(x)=\frac{1}{(b-a)^n}\binom{n}{i}(x-a)^n(b-x)^{n-i};\quad i=0,1,\ldots,n.$$

For convenience, we set $p_{i,n}(x) = 0$, if $i < 0$ or $i > n$.

We give the properties of the generalized Bernstein basis polynomials the following list:

(a) Positivity property:

$p_{i,n}(x) > 0$ is hold for all i=0,1,...,n and all $x \in [a,b]$.

(b) Unity partition property:

$$\sum_{i=0}^{n} p_{i,n}(x) = \sum_{i=0}^{n-1} p_{i,n-1}(x) = \ldots = \sum_{i=0}^{1} p_{i,1}(x) = 1.$$

(c) Recursion's relation property:

$$p_{i,n}(x) = \frac{1}{b-a}\left[(b-x)\,p_{i,n-1}(x) + (x-a)\,p_{i-1,n-1}(x)\right].$$

(d) First derivatives of the generalized Bernstein basis polynomials:

$$\frac{d}{dx} p_{i,n}(x) = \frac{n}{b-a}\left[p_{i-1,n-1}(x) + p_{i,n-1}(x)\right].$$

Definition 1.2 Let $y : [a,b] \to \mathbb{R}$ be continuous function on the interval [a, b]. Bernstein polynomials of nth-degree are defined by

$$B_n(y;x) = \sum_{i=0}^{n} y\left(a + \frac{(b-a)i}{n}\right) p_{i,n}(x).$$

Theorem 1.1 If $y \in C^k[a,b]$, for some integer $m \geq 0$, then

$$\lim_{n\to\infty} B_n^{(k)}(y;x) = y^{(k)}(x); \quad k = 0,1,\ldots,m$$

converges uniformly.

For more information about Bernstein polynomials, see [6, 7].

Consider the mth-order nonlinear differential equation

$$y_m(x) = f\left(x, y(x), y'(x), \ldots, y^{(m-1)}(x)\right), \quad a \leq x \leq b, \tag{1}$$

under the initial conditions

$$\sum_{k=0}^{m-1} \lambda_{jk} y^{(k)}(c) = \mu_j; \quad j = 0,1,\ldots,m-1, \; c \in [a,b]; \tag{2}$$

or boundary conditions

$$\sum_{k=0}^{m-1}\left[\alpha_{jk} y^{(k)}(a) + \beta_{jk} y^{(k)}(b)\right] = \gamma_j; \quad j = 0,1,\ldots,m-1. \tag{3}$$

Here f is nonlinear function and $f_{y^{(k)}} = \dfrac{\partial f}{\partial y^{(k)}}$ is functional derivatives of the $f\left(x, y(x), y'(x), \ldots, y^{(m-1)}(x)\right)$ on the interval [a, b], α_{jk}, β_{jk}, λ_{jk}, μ_j and γ_j are known constants, and $y(x)$ is unknown function.

In this paper, first purpose is to express the nonlinear equation (1) with conditions (2) or (3) as a sequence of mth-order linear differential equations by using quasilinearization method [9] iteratively:

$$y_{r+1}^{(m)} = f\left(x, y_r, y'_{r+1}, \ldots, y_{r+1}^{(m-1)}\right) + \sum_{k=0}^{m-1}\left(y_{r+1}^{(k)} - y_r^{(k)}\right) f_{y^{(k)}}\left(x, y_r, y'_r, \ldots, y_r^{(m-1)}\right). \quad (4)$$

under the initial conditions

$$\sum_{k=0}^{m-1} \lambda_{ij} y_{r+1}^{(k)}(c) = \mu_j; j = 0,1,\ldots,m-1, \quad c \in [a,b] \quad (5)$$

or boundary conditions

$$\sum_{k=0}^{m-1}\left[\alpha_{jk} y_{r+1}^{(k)}(a) + \beta_{jk} y_{r+1}^{(k)}(b)\right] = \gamma_j; \quad j = 0,1,\ldots,m-1. \quad (6)$$

Second purpose is to approximate the solutions of linear differential equations (4) with Bernstein polynomials:

$$y_{r+1}^{(k)}(x) \cong B_n^{(k)}\left(y_{r+1}; x\right) = \sum_{i=0}^{n} y_{r+1}\left(a + \tfrac{(b-a)i}{n}\right) p_{i,n}(x). \quad (7)$$

The paper is organized as follows. In Section 2, some fundamental relations are given for the generalized Bernstein basis polynomials and its derivatives. Combining the quasilinearization and the Bernstein collocation methods, the approximation solutions of the nonlinear differential equations are introduced in Section 3. In Section 4, some numerical examples are presented for exhibiting the accuracy and applicability of the proposed method. The Section 5 is ended with the conclusions.

2. FUNDAMENTAL RELATIONS

Theorem 2.1 Any generalized Bernstein basis polynomials of degree *n* can be written as a linear combination of the generalized Bernstein basis polynomials of degree *n* + 1:

$$p_{i,n}(x) = \frac{n-i+1}{n+1} p_{i,n}(x) + \frac{i+1}{n+1} p_{i+1,n+1}(x).$$

Proof. We can easily prove this theorem via definition of the generalized Bernstein polynomials. For more information, see [6].

Theorem 2.2 The first derivatives of *n*th degree generalized Bernstein basis polynomials can be written as a linear combination of the generalized Bernstein basis polynomials of degree *n*:

$$\frac{d}{dx} p_{i,n}(x) = \frac{1}{b-a}\left[(n-i+1) p_{i-1,n}(x) + (2i-n) p_{i,n}(x) - (i+1) p_{i+1,n}(x)\right].$$

Proof. By utilizing Theorem 2.1, the following functions can be written as

$$p_{i,n}(x) = \frac{n-i+1}{n} p_{i-1,n}(x) + \frac{i}{n} p_{i,n}(x),$$

$$p_{i-1,n-1}(x) = \frac{n-i+1}{n} p_{i-1,n}(x) + \frac{i}{n} p_{i,n}(x).$$

Substituting these relations in to the right hand side of the property (d), the desired relation is obtained.

Theorem 2.3 There is a relation between generalized Bernstein basis polynomials matrix and their derivatives in the form

$$\mathbf{P}^{(k)}(x) = \mathbf{P}(x)\,\mathbf{N}^k; \quad k = 1,\ldots,n.$$

Here the elements of $(n + 1) \times (n +1)$ matrix $\mathbf{N} = (m_{ij})$, $i, j = 0,1,\ldots,n$ are defined by:

$$m_{ij} = \frac{1}{b-a}\begin{cases} n-i, & \text{if} \quad j=i+1 \\ 2i-n, & \text{if} \quad j=i \\ -i, & \text{if} \quad j=i-1 \\ 0, & \text{otherwise} \end{cases}.$$

Proof. From Theorem 2.2 and condition, $p_{i,n}(x) = 0$ if $i < 0$ or $i > n$, we have

$$p'_{0,n}(x) = \frac{1}{b-a}\left[-np_{0,n}(x) - p_{1,n}(x)\right]$$
$$p'_{1,n}(x) = \frac{1}{b-a}\left[np_{0,n}(x) + (2-n)p_{1,n}(x) - 2p_{2,n}(x)\right]$$
$$p'_{2,n}(x) = \frac{1}{b-a}\left[(n-1)p_{1,n}(x) + (4-n)p_{2,n}(x) - 3p_{3,n}(x)\right]$$
$$\vdots$$
$$p'_{n-1,n}(x) = \frac{1}{b-a}\left[2p_{n-2,n}(x) + (n-2)p_{n-1,n}(x) - np_{n,n}(x)\right]$$
$$p'_{n,n}(x) = \frac{1}{b-a}\left[p_{n-1,n}(x) + np_{n,n}(x)\right].$$

Hence we obtain the matrix relation

$$\mathbf{P}'(x) = \mathbf{P}(x)\mathbf{N}$$

such that

$$\mathbf{P}(x) = \left[p_{0,n}(x) \quad p_{1,n}(x) \quad \ldots \quad p_{n,n}(x)\right],$$
$$\mathbf{P}'(x) = \left[p'_{0,n}(x) \quad p'_{1,n}(x) \quad \ldots \quad p'_{n,n}(x)\right],$$

$$\mathbf{N} = \begin{bmatrix} -n & n & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2-n & n-1 & \ldots & 0 & 0 & 0 \\ 0 & -2 & 4-n & \ldots & 0 & 0 & 0 \\ 0 & 0 & -3 & \ldots & 0 & 0 & 0 \\ \vdots & & & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & n-4 & 2 & 0 \\ 0 & 0 & 0 & \ldots & 1-n & n-2 & 1 \\ 0 & 0 & 0 & \ldots & 0 & -n & n \end{bmatrix}.$$

In a similar way, the second derivatives

$$\mathbf{P}''(x) = \mathbf{P}'(x)\mathbf{N} = \mathbf{P}(x)\mathbf{N}^2.$$

Thus we get derivatives of the unknown function in the form

$$\mathbf{P}^{(k)}(x) = \mathbf{P}^{(k-1)}(x)\mathbf{N} = \mathbf{P}(x)\mathbf{N}^k.$$

This completes the proof.

3. METHOD OF THE SOLUTION

Theorem 3.1 Let $x_i \in [a,b]$; $i = 0,1,\ldots,n$ be collocation points. General mth-order nonlinear differential equation (1) can be written as the matrix form of a sequence of linear differential equations:

$$\left[\mathbf{PN}^m - \sum_{k=0}^{m-1}\mathbf{H}_{r+1}\mathbf{PN}^k\right]\mathbf{Y}_{r+1} = \mathbf{G}_{r+1}; r = 0,1... \tag{8}$$

Here the matrices are $\mathbf{H}_{r+1} = diag\left[h_{r+1}\left(x_i\right)\right]$, $\mathbf{P} = \left\lfloor p_{j,n}\left(x_i\right)\right\rfloor$, $\mathbf{G}_{r+1} = \left[g_{r+1}\left(x_i\right)\right]$ and $\mathbf{Y}_{r+1} = \left[y_{r+1}\left(a + \frac{(b-a)i}{n}\right)\right];\ i, j = 0,...,n.$

Proof. Let $y_0\left(x\right)$ be chosen function that provide given initial or boundary conditions. Consider the sequence of linear differential equations (4) for nonlinear differential equation (1) as follows:

$$y_{r+1}^{(m)} = \sum_{k=0}^{m-1} f_{y^{(k)}}\left(x, y_r, y_r', ..., y_r^{(m-1)}\right) y_{r+1}^{(k)} - f_{y^{(k)}}\left(x, y_r, y_r', ..., y_r^{(m-1)}\right) y_r^{(k)} + f\left(x, y_r, y_r', ..., y_r^{(m-1)}\right).$$

We use the Bernstein collocation method for solving these series of linear equations. The expression (7) can be denoted by the matrix form

$$y_{r+1}^{(k)}(x) \cong B_n^{(k)}\left(y_{r+1}; x\right) = \mathbf{P}^{(k)}\mathbf{Y}_{r+1}.$$

By utilizing Theorem 2.3, the derivatives of the unknown functions can also be written by

$$y_{r+1}^{(k)}(x) \cong \mathbf{P}\left(x\right)\mathbf{N}^k\mathbf{Y}_{r+1};\quad k = 0,1,...,m. \tag{9}$$

Substituting the collocation points and relation (9) into equations (4), we obtain the linear algebraic equation systems

$$\mathbf{P}\left(x_i\right)\mathbf{N}^m\mathbf{Y}_{r+1} - \sum_{k=0}^{m-1} h_{r+1}\left(x_i\right)\mathbf{P}\left(x_i\right)\mathbf{N}^k\mathbf{Y}_{r+1} = g_{r+1}\left(x_i\right);\quad i = 0,\ldots,n \tag{10}$$

such that $y_{r+1}^{(k)}(x_i) = B_n^{(k)}\left(y_{r+1}; x_i\right);\ k = 0,1,\ldots,m$. Here $h_{r+1}\left(x_i\right)$ and $g_{r+1}\left(x_i\right)$ is denoted by

$$h_{r+1}\left(x_i\right) = f_{y^{(k)}}\left(x_i, y_r\left(x_i\right), y_r'\left(x_i\right), ..., y_r^{(m-1)}\left(x_i\right)\right),$$

$$g_{r+1}\left(x_i\right) = f\left(x_i, y_r\left(x_i\right), y_r'\left(x_i\right), ..., y_r^{(m-1)}\left(x_i\right)\right)$$
$$- \sum_{k=0}^{m-1} f_{y^{(k)}}\left(x_i, y_r\left(x_i\right), y_r'\left(x_i\right), ..., y_r^{(m-1)}\left(x_i\right)\right) y_r^{(k)}\left(x_i\right).$$

Considering the matrices

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}(x_0) \\ \mathbf{P}(x_1) \\ \vdots \\ \mathbf{P}(x_n) \end{bmatrix},\ \mathbf{H}_{r+1} = \begin{bmatrix} h_{r+1}(x_0) & 0 & \ldots & 0 \\ 0 & h_{r+1}(x_1) & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & h_{r+1}(x_n) \end{bmatrix},\ \mathbf{G}_{r+1} = \begin{bmatrix} g_{r+1}(x_0) \\ g_{r+1}(x_1) \\ \vdots \\ g_{r+1}(x_n) \end{bmatrix},$$

the linear equation systems (10) can be denoted by the matrix form (8) and the proof is completed.

We can solve the differential equation with variable coefficients under the conditions given in the following steps:

Step 1. The equation (8) can be written in the compact form

$$\mathbf{W}_{r+1}\mathbf{Y}_{r+1} = \mathbf{G}_{r+1} \text{ or } \left[\mathbf{W}_{r+1}; \mathbf{G}_{r+1}\right],$$

so that $\mathbf{W}_{r+1} = \mathbf{PN}^m - \sum_{k=0}^{m-1} \mathbf{H}_{r+1}\mathbf{PN}^k$. This matrix equation (11) corresponds to a linear algebraic systems with unknown coefficients $y_{r+1}; r = 0,1,\ldots$ iteratively.

Step 2. From the expression (9), the matrix forms of the conditions (5) and (6) can be written respectively

$$\mathbf{V}_j = \sum_{k=0}^{m-1} \lambda_{jk}\mathbf{P}(c)\mathbf{N}^k = \begin{bmatrix} v_{j,0} & v_{j,1} & \cdots & v_{j,n} \end{bmatrix},$$

$$\mathbf{U}_j = \sum_{k=0}^{m-1} [\alpha_{jk}\mathbf{P}(a)\mathbf{N}^k + \beta_{jk}\mathbf{P}(b)\mathbf{N}^k] = \begin{bmatrix} u_{j,0} & u_{j,1} & \cdots & u_{j,n} \end{bmatrix}$$

or compactly

$$\mathbf{V}_j\mathbf{Y}_{r+1} = \mu_{\mathbf{j}} \quad \text{or} \quad [\mathbf{V}_j; \mu_j],$$

$$\mathbf{U}_j\mathbf{Y}_{r+1} = \gamma_{\mathbf{j}} \quad \text{or} \quad [\mathbf{U}_j; \gamma_j].$$

Step 3. To obtain the solutions of equations (4) under the conditions (5) or (6), we add the elements of the row matrices (12) or (13) to the end of the matrix (11). In this way, we have the new augmented matrix $\left[\tilde{\mathbf{W}}_{r+1}; \tilde{\mathbf{G}}_{r+1}\right]$. Here the augmented matrix is a $(n+m+1)\times(n+1)$ rectangular matrix. This new matrix equation shortly can be denoted by $\tilde{\mathbf{W}}_{r+1}\mathbf{Y}_{r+1} = \tilde{\mathbf{G}}_{r+1}$.

Step 4. If $rank\left(\tilde{\mathbf{W}}_{r+1}\right) = rank\left[\tilde{\mathbf{W}}_{r+1}; \tilde{\mathbf{G}}_{r+1}\right] = n+1$, then unknown coefficients $y_{r+1}; r = 0,1,\ldots$ are uniquely determined iteratively. These kinds of systems can be solved by the Gauss Elimination, Generalized Inverse and QR factorization methods.

4. NUMERICAL RESULTS

Two numerical examples are considered by using the presented method on collocation points $x_i = a + \frac{(b-a)i}{n}; i = 0,1,\ldots n$. Numerical results are written in Matlab 7.1.

Example 4.1 Consider the following nonlinear boundary value problem [2]:

$$y'' = e^y;\ 0 < \text{x} < 1, y(0) = y(1) = 0.$$

The exact solution of the above equation is $y(x) = -\,ln2 + 2ln[c \sec(c(x - 1/2)/2)]$, where $c = 1.3360557$. Let be $y_0(x) = 0$.

Table 1. Maximum errors of Example 4.1.

n	E_1	E_2	E_3
2	1.0e - 002	1.3e - 002	1.3e - 002
4	5.3e - 004	1.4e - 005	1.4e - 005
8	5.2e - 004	8.5e - 009	9.6e - 009
16	5.2e - 004	8.5e - 009	8.5e - 009
32	5.2e - 004	8.5e - 009	8.5e - 009

Using the Bernstein collocation method, the maximum errors are given in the Table 1. The numerical results show that the proposed method can be applicable to the nonlinear differential equations and have effective results for increasing *n*.

Example 4.2 Consider the following nonlinear boundary value problem:

$$y'' = 2y^3;\ x \in [0,1];\quad y(0)=1,\ y(1)=1/2$$

The analytic solution of the above equation is $y(x) = 1/(1+x)$. Let be $y_0(x) = 1 - x/2$.

Table 2. Absolute errors of Example 4.2.

n	E_1	E_2	E_3	E_4
4	6.4e - 002	6.3e - 002	6.4e – 002	6.4e - 002
6	2.2e - 003	1.5e - 006	1.4e – 006	1.4e - 006
16	2.2e - 003	1.3e - 006	8.1e – 012	8.2e - 012
24	2.2e - 003	1.3e - 006	4.6e – 013	4.9e - 015

In Table 2, the maximum errors are computed with increasing *n*. We show that the presented method converges rapidly to the exact solution of the nonlinear differential equations for increasing *n*.

5. CONCLUSIONS

In this study, by using quasilinearization technique, nonlinear differential equations under the initial or boundary conditions are expressed as a sequence of the linear differential equations iteratively. Then, a collocation method based on generalized Bernstein polynomials is developed for solving these equations. If $y(x)$ and its derivatives are continuous functions on bounded interval $[a,b]$, then the Bernstein collocation method can be applied to any initial or boundary value problems. Moreover, this method has been tested on two nonlinear boundary problems, and numerical results have been presented for showing applicability, accuracy of the proposed method. Consequently, all of the reasons are revealed that the proposed method is encouraging for solutions of the other problems involving different equations.

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