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APPROXIMATION OF CONJUGATE FUNCTIONS BY TRIGONOMETRIC POLYNOMIALS IN WEIGHTED ORLICZ SPACES

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Abstract. We investigate the approximation of a conjugate function by the Fejér sums of the Fourier series of the conjugate function and obtain the estimate between the derivatives of the conjugate functions and the derivatives of the conjugate trigonometric polynomials in the weighted Orlicz spaces with Muckenhoupt weights. We prove inverse theorem of approximation theory for the derivatives conjugate functions in the weighted Orlicz spaces.

1. Introduction and new results

A continuous and convex function $M:[0,\infty)\to[0,\infty)$ which satisfies the conditions

$$M(0)=0; \qquad M>0 \ \text{ for } \ x>0,$$

$$\lim_{x\to 0}\frac{M(x)}{x}=0; \qquad \lim_{x\to\infty}\frac{M(x)}{x}=\infty,$$

is called an $N-$*function*. The *complementary* $N-$*function to* M is defined by

$$N(y):=\max_{x\geqslant 0}\left(xy-M(x)\right)$$

for $y>0$ [35, p. 11].

We denote by T the interval $[-\pi,\pi]$ and $\mathbb{C}$ the complex plane. Let M be an $N-$function and N be its complementary function. By $L_M(T)$ we denote the linear space of Lebesgue measurable functions $f:T\to\mathbb{C}$ satisfying the condition

$$\int_T M\left(\alpha|f(x)|\right)dx<\infty$$

for some $\alpha>0$, equipped with the norm

$$\|f\|_{L_M(T)}:=\sup\left\{\int_T |f(x)\cdot g(x)|\,dx:\ g\in L_N(T),\ \rho(g,N)\leqslant 1\right\},$$

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where

$$\rho(g,N) := \int_T N(|g(x)|\,dx).$$

The space $L_M(T)$ is a Banach space [42, pp. 52–68]. The norm $\|\cdot\|_{L_M(T)}$ is called *Orlics norm* and the space $L_M(T)$ is called *Orlicz space*.

Note that the Orlicz spaces are known as the generalizations of the Lebesgue space $L_p(T)$, $1<p<\infty$.

A function ω is called a *weight* on T if $\omega : T \to [0,\infty]$ is a measurable and $\omega^{-1}(\{0,\infty\})$ has measure zero (with respect to Lebesgue measure).

The class of measurable functions f defined on T and satisfying the condition $\omega f \in L_M(T)$ is called *weighted Orlicz space* $L_M(T,\omega)$ with the norm

$$\|f\|_{L_M(T,\omega)} := \|f\omega\|_{L_M(T)}.$$

Let $M^{-1} : [0,\infty) \to [0,\infty)$ be the inverse function of the $N-$function M. The *lower* and *upper Boyd indices* α_M and β_M are defined by

$$\alpha_M = \lim_{t\to+\infty} \theta(t) = \sup_{t>1} \theta(t), \qquad \beta_M = \lim_{t\to 0^+} \theta(t) = \inf_{0<t<1} \theta(t),$$

where $\theta(t) = -\log h(t)/\log t$, and for Orlicz spaces [6], [9], [36]

$$h(t) = \lim_{x\to\infty} \sup \frac{M^{-1}(x)}{M^{-1}(tx)}, \; t>0.$$

The Boyd indices α_M, β_M are known to be *nontrivial* if $0<\alpha_M$ and $\beta_M<1$. It is known that

$$0 \leqslant \alpha_M \leqslant \beta_M \leqslant 1$$

and

$$\alpha_N + \beta_M = 1, \qquad \alpha_M + \beta_N = 1.$$

The space $L_M(T)$ is reflexive if and only if $0<\alpha_M \leqslant \beta_M < 1$.

Let $1<p<\infty$, $1/p+1/q$. A weight function ω belongs to the *Muckenhoupt class* $A_p(T)$ if

$$\left(\frac{1}{|I|}\int_I \omega^p(x)\,dx\right)^{1/p} \left(\frac{1}{|I|}\int_I \omega^{-q}(x)\,dx\right)^{1/q} \leqslant C,$$

with a finite C independent of I, where I is any subinterval of T and $|I|$ denotes the length of I.

Note that the weight functions belong to the class A_p, introduced by Muckenhoupt [37], play a very important role in different fields of mathematical analysis.

Let $L_M(T,\omega)$ be a weighted Orlicz space with Boyd indices $0<\alpha_M \leqslant \beta_M < 1$, and let $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. For $f \in L_M(T,\omega)$ the shift operator can be defined as:

$$f_h(x) := \frac{1}{2h}\int_{-h}^{h} f(x+t)\,dt, \quad 0<h<\pi, \quad x \in T.$$

The function

$$\Omega^k_{M,\omega}(\delta,f) := \sup_{\substack{0<h_i\leqslant\delta \\ 1\leqslant i\leqslant k}} \left\| \prod_{i=1}^{k} \left(I - f_{h_i}\right) f \right\|_{L_M(T,\omega)}, \quad \delta > 0, \quad k = 1,2,\ldots$$

is called $k-th$ *modulus of smoothness of* g, where I is identity operator. It is known [26] that f_h is a bounded linear operator on $L_M(T,\omega)$. If $k=0$ we set $\Omega^0_{M,\omega}(\delta,g) := \|g\|_{L_M(T,\omega)}$ and $k=1$ we write $\Omega_{M,\omega}(\delta,g) := \Omega^1_{M,\omega}(\delta,g)$.

The *function conjugate* to a 2π-periodic summable function on $[-\pi,\pi]$ given by

$$\widetilde{f}(x) = \lim_{\varepsilon\to 0+}\left\{-\frac{1}{\pi}\int_{\varepsilon}^{\pi}\frac{f(x+t)-f(x-t)}{2\tan\frac{t}{2}}dt\right\} = -\frac{1}{\pi}\int_{0}^{\pi}\frac{f(x+t)-f(x-t)}{2\tan\frac{t}{2}}dt$$

exists almost-everywhere.

Let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} A_k(x,f), \qquad A_k(x,f) := a_k \cos kx + b_k \sin kx \tag{1.1}$$

be the *Fourier series* of the function $f \in L_1(T)$. Then in the case where the *conjugate trigonometric series*

$$-i\sum_{k=-\infty}^{\infty} \text{sign } kc_k e^{ikx} = \sum_{k=1}^{\infty}(a_k \sin kx - b_k \cos kx)$$

is the Fourier series of some function $\tilde{f}$. It is know that the conjugate series to Fourier series $f \in L_{[0,2\pi]}$ will not always be the Fourier series (see, for example, [47, p. 155]). The *nth partial sums, Fejér sums* of the series (1.1) are defined, respectively, as

$$S_n(x,f) = \frac{a_0}{2} + \sum_{k=1}^{n} A_k(x,f),$$

$$\sigma_n(f) = \frac{1}{n+1}\sum_{k=0}^{n} S_k(x,f).$$

For $f \in L_M(T,\omega)$ we define the derivative of f as a function g satisfying

$$\lim_{h\to 0}\left\|\frac{1}{h}(f(x+h)-f(x)) - g(x)\right\|_{L_M(T,\omega)} = 0.$$

in which case we write $g = f'$. Then we say that the function $f \in L_M(T,\omega)$ has *derivative in the sense* $L_M(T,\omega)$. Let

$$E_n(f)_{M,\omega} := \inf_{T_n\in\Pi_n}\|f - T_n\|_{L_M(T,\omega)}$$

be the best approximation to $f \in L_M(T,\omega)$ in the class Π_n of trigonometric polynomials of degree not greater than n. Note that the existence of $T_n^* \in \Pi_n$ such that

$$E_n(f)_{M,\omega} := \|f - T_n^*\|_{L_M(T,\omega)}$$

follows, for example, from Theorem 1.1 in [11, p. 59].

Note that the problems of existence of the derivative of function and approximation of the function, conjugate function , its derivative by polynomials and rational function in different spaces are investigated by several authors (see, for example, [1–8], [10–34], [38–41], [43–55], etc.).

In the present paper we investigate the approximation of a conjugate function by the Fejér sums of the Fourier series of the conjugate function in the weighted Orlicz spaces $L_M(T,\omega)$. Under certain conditions, we obtain the estimate between the derivatives of the conjugate functions and the derivatives of the conjugate trigonometric polynomials in the weighted Orlicz space $L_M(T,\omega)$. Note that the estimate obtained between the derivatives of the conjugate functions and the derivatives of the conjugate trigonometric polynomials depends on sequence of the best approximation in the weighted Orlicz spaces $L_M(T,\omega)$.

In addition, we obtain inverse theorem of approximation theory for the derivatives conjugate functions in the weighted Orlicz spaces $L_M(T,\omega)$.

We use c_1, c_2,... to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the questions of interest.

Our main results are the following.

THEOREM 1.1. *Let $L_M(T,\omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \leqslant \beta_M < 1$ and let $f \in L_M(T,\omega)$, $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. Then $\tilde{f}^{(r)} \in L_M(T,\omega)$ and the estimate*

$$\left\|\widetilde{f}(x) - \sigma_{n-1}\left(\widetilde{f}\right)\right\|_{L_M(T,\omega)} \leqslant c_1\left(\Omega_{M,\omega}\left(\frac{1}{n+1}, f\right) + E_{n+1}\left(\widetilde{f}\right)_{M,\omega}\right), \quad (n = 1,2,...)$$

holds with a constant $c_1 > 0$ independent of n.

THEOREM 1.2. *Let $L_M(T,\omega)$ be a weighted Orlicz space with Boyd indices $0 < \alpha_M \leqslant \beta_M < 1$ and let $f \in L_M(T,\omega)$, $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. Then $\tilde{f}^{(r)} \in L_M(T,\omega)$ and if T_n is the best approximation trigonometric polynomial to f in the space $L_M(T,\omega)$ and for some natural r satisies the condition*

$$\sum_{n=1}^{\infty} n^{r-1} E_n(f)_{M,\omega} < \infty, \tag{1.2}$$

then

$$\left\|\tilde{f}^{(r)} - \tilde{T}_n^{(r)}\right\|_{L_M(T,\omega)} \leqslant c_2 \left\{ n^r E_n(f)_{M,\omega} + \left(\sum_{\mu=n+1}^{\infty} \mu^{r-1} E_\mu(f)_{M,\omega} \right) \right\}.$$

THEOREM 1.3. *Let $L_M(T,\omega)$ be a weighted Orliez space with Boyd indices $0 < \alpha_M \leqslant \beta_M < 1$, and let $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}(T)$. If $f \in L_M(T,\omega)$ satisfies, for some natural r,*

$$\sum_{n=1}^{\infty} n^{r-1} E_n(f)_{M,\omega} < \infty,$$

then $\tilde{f}^{(r)} \in L_M(T,\omega)$ and for every natural number n the estimate

$$\Omega_{M,\omega}^{k}\left(\frac{1}{n},\tilde{f}^{(r)}\right) \leqslant c_3 \left\{\frac{1}{n^{2k}}E_0(f)_{M,\omega}+\frac{1}{n^{2k}}\sum_{q=1}^{n} q^{2k+r-1}E_q(f)_{M,\omega}+\sum_{q=n+1}^{\infty} q^{r-1}E_q(f)_{M,\omega}\right\},$$
$$k = 1,2,...,$$

holds with a constant c_3 independent of n.

2. Proofs of the new results

Proof of Theorem 1.1. We set

$$T_{2n}(x) = \frac{1}{n+1}\sum_{k=n}^{2n} S_k(x,f),$$

where $S_k(x,f)$ is the n-th partial sums of the function $f \in L_1(T)$.

According to [20] and [26]

$$\|f(x) - T_{2n}(x)\|_{L_M(T,\omega)} \leqslant c_4 E_{n+1}(f)_{M,\omega}. \tag{2.1}$$

By [18] for any function $f \in L_M(T,\omega)$ and $g \in L_M(T,\omega)$ we get

$$\left\|\tilde{f}\right\|_{L_M(T,\omega)} \leqslant c_5 \|f\|_{L_M(T,\omega)}, \qquad \|\sigma_{n-1}(g)\|_{L_M(T,\omega)} \leqslant c_6 \|g\|_{L_M(T,\omega)}. \tag{2.2}$$

Then taking into account (2.2) and the triange inequality, we obtain

$$\begin{aligned}\|g - \sigma_{n-1}(g)\|_{L_M(T,\omega)} &= \|g - T_{2n}(g) + T_{2n}(g) - \sigma_{n-1}(g)\|_{L_M(T,\omega)} \\ &\leqslant \|g - T_{2n}(g)\|_{L_M(T,\omega)} + \|T_{2n}(g) - \sigma_{n-1}(g)\|_{L_M(T,\omega)} \\ &= \|g - T_{2n}(g)\|_{L_M(T,\omega)} + \|\sigma_{n-1}(T_{2n}(g) - g)\|_{L_M(T,\omega)} \\ &\leqslant \|g - T_{2n}(g)\|_{L_M(T,\omega)} + c_7 \|g - T_{2n}(g)\|_{L_M(T,\omega)} \\ &\leqslant (1 + c_8)\|g - T_{2n}(g)\|_{L_M(T,\omega)}.\end{aligned} \tag{2.3}$$

According to (2.1) and (2.3), we have

$$\|g - \sigma_{n-1}(g)\|_{L_M(T,\omega)} \leqslant c_9 E_{n+1}(g)_{M,\omega}.$$

Now in this inequality assuming $g = \tilde{f}$

$$\left\|\tilde{f} - \sigma_{n-1}(\tilde{f})\right\|_{L_M(T,\omega)} \leqslant c_{10} E_{n+1}(\tilde{f})_{M,\omega}. \tag{2.4}$$

It is known from [29] that

$$E_n(f)_{M,\omega} \leqslant c_{11}\Omega^k_{M,\omega}\left(\frac{1}{n+1}, f\right) \tag{2.5}$$

By virtue of (2.4) and (2.5) we obtain

$$\left\|\tilde{f} - \sigma_{n-1}(\tilde{f})\right\|_{L_M(T,\omega)} \leqslant c_{12}\left\{\Omega_{M,\omega}\left(\frac{1}{n+1}, f\right) + E_{n+1}(\tilde{f})_{M,\omega}\right\}.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. From the condition (1.2), it follows $f^{(r)} \in L_M(T,\omega)$. Consequently, due to boundedness of adjoint operator we obtain $\tilde{f}^{(r)} \in L_M(T,\omega)$. For the natural number n we consider trigonometric polynomial $T_{2^i n}$, where $i = 0, 1, \ldots$. The following relation holds:

$$f(x) = T_n(x) + \sum_{i=0}^{\infty}(T_{2^{i+1}n}(x) - T_{2^i n}(x)). \tag{2.6}$$

Since trigonometric polynomial T_n is the polynomial of best approximation to f, the right side of the series (2.6) converges by the norm of space $L_M(T,\omega)$. Then by (2.6), for every natural number n, we obtain

$$\begin{aligned}\|T_{2^{i+1}n}(x) - T_{2^i n}(x)\|_{L_M(T,\omega)} &\leqslant \|T_{2^{i+1}n}(x) - f(x)\|_{L_M(T,\omega)} + \|f(x) - T_{2^i n}(x)\|_{L_M(T,\omega)} \\ &\leqslant E_{2^{i+1}n}(f)_{M,\omega} + E_{2^i n}(f)_{M,\omega} \leqslant 2E_{2^i n}(f)_{M,\omega}.\end{aligned}$$

It is clear that

$$T_{2^{i+1}n}(x) - T_{2^i n}(x)$$

is trigonometric polynomial of degree at most $2^{i+1}n$. Since $E_n(\tilde{f})_{M,\omega} \leqslant cE_n(f)_{M,\omega}$ [26] and the sequence $\{E_n(f)_{M,\omega}\}$ is monotone, then using the Bernstein inequality for weighted Orlicz spaces [26] we have

$$\begin{aligned}\left\|\tilde{T}^{(r)}_{2^{i+1}n}(x) - \tilde{T}^{(r)}_{2^i n}(x)\right\|_{L_M(T,\omega)} &\leqslant (2^{i+1}n)^r \left\|\tilde{T}_{2^{i+1}n}(x) - \tilde{T}_{2^i n}(x)\right\|_{L_M(T,\omega)} \\ &\leqslant (2^{i+1}n)^r\left(\left\|\tilde{f} - \tilde{T}_{2^{i+1}n}\right\|_{L_M(T,\omega)} + \left\|\tilde{f} - \tilde{T}_{2^i n}\right\|_{L_M(T,\omega)}\right) \\ &\leqslant (2^{i+1}n)^r\left(E_{2^{i+1}n}(\tilde{f})_{M,\omega} + E_{2^i n}(\tilde{f})_{M,\omega}\right) \\ &\leqslant (2^{i+1}n)^r 2E_{2^i n}(f)_{M,\omega} \\ &\leqslant 2^{r+1}(2^i n)^r E_{2^i n}(f)_{M,\omega}.\end{aligned} \tag{2.7}$$

Let

$$\tilde{T}_n(x) + \sum_{i=0}^{\infty}(\tilde{T}_{2^{i+1}n}(x) - \tilde{T}_{2^i n}(x))$$

be the conjugate series of the series (2.6).

Then, taking into account (2.7) we get

$$\left\|\tilde{T}_n^r(x)+\sum_{i=0}^{\infty}(\tilde{T}_{2^{i+1}n}^{(r)}(x)-\tilde{T}_{2^in}^{(r)}(x)\right\|_{L_M(T,\omega)} \leqslant c_{13}2^{r+1}\sum_{i=0}^{\infty}(2^in)^rE_{2^in}(f)_{M,\omega}. \tag{2.8}$$

Since the condition (1.2) holds, then we have the following inequality

$$\begin{aligned}\sum_{m=n+1}^{\infty} m^{r-1}E_m(f)_{M,\omega} &= \sum_{i=0}^{\infty}\sum_{m=2^in+1}^{2^{i+1}n} m^{r-1}E_m(f)_{M,\omega}\\ &\geqslant \sum_{i=0}^{\infty}(2^in)^{r-1}E_{2^{i+1}n}(f)_{M,\omega}2^in = \sum_{i=0}^{\infty}(2^in)^rE_{2^{i+1}n}(f)_{M,\omega}.\end{aligned}$$

Then

$$\begin{aligned}\sum_{i=0}^{\infty}(2^in)^rE_{2^in}(f)_{M,\omega} &= n^rE_n(f)_{M,\omega}+\sum_{i=0}^{\infty}(2^{i+1}n)^rE_{2^{i+1}n}(f)_{M,\omega}\\ &\leqslant n^rE_n(f)_{M,\omega}+2^r\sum_{m=n+1}^{\infty} m^{r-1}E_n(f)_{M,\omega}.\end{aligned}$$

The last inequality yields

$$2^{r+1}\sum_{i=0}^{\infty}(2^in)^rE_{2^in}(f)_{M,\omega} \leqslant c_{14}\{n^rE_n(f)_{M,\omega}+\sum_{m=n+1}^{\infty} m^{r-1}E_n(f)_{M,\omega}\}. \tag{2.9}$$

According to (2.8) and (2.9) the series

$$\tilde{T}_n^{(r)}(x)+\sum_{i=0}^{\infty}\left(\tilde{T}_{2^{i+1}n}^{(r)}(x)-\tilde{T}_{2^in}^{(r)}(x)\right) \tag{2.10}$$

converges by the norm of space $L_M(T,\omega)$ to some function. It is clear that for the derivative $\tilde{f}^{(r)}$ in the sense $L_M(T,\omega)$

$$\tilde{f}^{(r)}(x) = \tilde{T}_n^{(r)}(x)+\sum_{i=0}^{\infty}(\tilde{T}_{2^{i+1}n}^{(r)}(x)-\tilde{T}_{2^in}^{(r)}(x)).$$

Using (2.9) and (2.10) we have

$$\begin{aligned}\left\|\tilde{f}^{(r)}(x)-\tilde{T}_n^{(r)}(x)\right\|_{L_M(T,\omega)} &\leqslant 2^{r+1}\sum_{i=0}^{\infty}(2^in)^rE_{2^in}(f)_{M,\omega}\\ &\leqslant c_{15}\{n^rE_n(f)_{M,\omega}+\sum_{m=n+1}^{\infty} m^{r-1}E_m(f)_{M,\omega}\}.\end{aligned}$$

The proof of Theorem 1.2 is completed. □

Proof of Theorem 1.3. According to Theorem 1.2 $\tilde{f}^{(r)} \in L_M(T,\omega)$, we have

$$E_n(\tilde{f}^{(r)})_{M,\omega} \leqslant c_{16}\left\{n^r E_n(f)_{M,\omega} + \left(\sum_{\mu=n+1}^{\infty} \mu^{r-1} E_\mu(f)_{M,\omega}\right)\right\}. \tag{2.11}$$

For the $k-$modulus of smoothness $\Omega^k_{M,\omega}(\cdot,f)$ the following estimate holds [26]:

$$\Omega^k_{M,\omega}\left(\frac{1}{n},f\right) \leqslant \frac{c_{17}}{n^{2k}}\left\{E_0(f)_{M,\omega} + \sum_{m=1}^{n} m^{2k-1} E_m(f)_{M,\omega}\right\}. \tag{2.12}$$

If the inequality (2.12) is applied to the function $\tilde{f}^{(r)}$, we get

$$\Omega^k_{M,\omega}\left(\frac{1}{n},\tilde{f}^{(r)}\right) \leqslant \frac{c_{18}}{n^{2k}}\left\{E_0\left(\tilde{f}^{(r)}\right)_{M,\omega} + \sum_{m=1}^{n} m^{2k-1} E_m\left(\tilde{f}^{(r)}\right)_{M,\omega}\right\}. \tag{2.13}$$

Using the estimates (2.11) and (2.13), we obtain

$$\begin{aligned}
&\Omega^k_{M,w}\left(\frac{1}{n},\tilde{f}^{(r)}\right)_{M,\omega} \\
&\leqslant \frac{c_{19}}{n^{2k}}\left\{E_0\left(\tilde{f}^{(r)}\right)_{M,\omega} + \sum_{m=1}^{n} m^{2k-1} E_m\left(\tilde{f}^{(r)}\right)_{M,\omega}\right\} \\
&\leqslant \frac{c_{20}}{n^{2k}}\left\{E_0(f)_{M,\omega} + \sum_{m=1}^{\infty} m^{r-1} E_m(f)_{M,\omega}\right. \\
&\qquad \left. + \sum_{m=1}^{n} m^{2k-1}\left[m^r E_m(f)_{M,\omega} + \sum_{p=m+1}^{\infty} p^{r-1} E_p(f)_{M,\omega}\right]\right\} \\
&\leqslant \frac{c_{21}}{n^{2k}}\left\{E_0(f)_{M,\omega} + \sum_{m=1}^{n} m^{2k+r-1} E_m(f)_{M,\omega} + \sum_{m=1}^{n} m^{2k-1} \sum_{p=m}^{\infty} m^{r-1} E_p(f)_{M,\omega}\right\} \\
&\leqslant \frac{c_{22}}{n^{2k}}\left\{E_0(f)_{M,\omega} + \sum_{m=1}^{n} m^{2k+r-1} E_m(f)_{M,\omega}\right. \\
&\qquad \left. + \sum_{m=1}^{n} m^{2k-1}\left[\sum_{p=m}^{n} p^{r-1} E_p(f)_{M,\omega} + \sum_{p=n+1}^{\infty} p^{r-1} E_p(f)_{M,\omega}\right]\right\} \\
&\leqslant c_{23}\left\{\frac{1}{n^{2k}} E_0(f)_{M,\omega} + \frac{1}{n^k}\sum_{m=1}^{n} m^{2k+r-1} E_m(f)_{M,\omega}\right. \\
&\qquad \left. + \frac{1}{n^{2k}}\sum_{p=1}^{n} p^{n-1} E_p(f)_{M,\omega} \sum_{m=1}^{p} m^{2k-1} + \sum_{p=n+1}^{\infty} p^{r-1} E_p(f)_{M,\omega}\right\}
\end{aligned}$$

$$\leqslant c_{24}\left\{\frac{1}{n^{2k}}E_0(f)_{M,\omega}+\frac{1}{n^{2k}}\sum_{m=1}^{n}m^{2k+r-1}E_m(f)_{M,\omega}+\frac{1}{n^{2k}}\sum_{p=1}^{n}p^{2k+r-1}E_p(f)_{M,\omega}\right\}$$

$$\leqslant c_{25}\left\{\frac{1}{n^{2k}}E_0(f)_{M,\omega}+\frac{1}{n^{2k}}\sum_{q=1}^{n}q^{2k+r-1}E_q(f)_{M,\omega}+\sum_{q=n+1}^{\infty}q^{r-1}E_q(f)_{M,\omega}\right\}.$$

The proof of Theorem 1.3 is completed. □

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REFERENCES

[1] V. V. ANDRIEVSKII AND D. M. ISRAFILOV, *Approximation of functions on quasiconformal curves by rational functions*, Izv. Akad. Nauk. Azer. SSR. Ser. Fiz. Tekhn. Math. Nauk, **4**, 1 (1980), 21–26.

[2] V. V. ANDRIEVSKII, *Approximation characterization of classes of functions on continua of the complex plane*, Math. Sb. (N.S), **125**, 167 (1984), 70–87.

[3] V. V. ANDRIEVSKII AND H.-P. BLATT, *Discrepancy of signed measures and polynomial approximation*, Springer-Verlag, New York, 2002.

[4] N. K. BARI AND S. B. STECHKIN, *Best approximation and differential properties of two conjugate functions*, Trudy Moskov. Matem. Obshestva, **5**, 1 (1956), 483–522.

[5] S. P. BAIBORODOV, *Approximation of conjugate functions by Fourier sums in $L^p_{2\pi}$*, Analysis Mathematica, **11**, 1 (1985), 3–12.

[6] C. BENNETT AND R. SHARPLEY, *Interpolation of Operators*, Academic Press, 1988.

[7] O. V. BESOV, *Interpolation of spaces of differentiable functions on a domain*, Trudy Mat. Inst.Steklova, **214**, 17 (1997), 59–82 (in Russian): translation in Proc. Steklov Ins. Math. **214**, 3 (1996), 54–76.

[8] O. V. BESOV AND G. KALYABIN, *Spaces of differentiable functions*, Function spaces, differential operators and nonlinear analysis, Birkhäuser, Basel, 2003, 3–21.

[9] D. W. BOYD, *Indices for the Orlicz spaces*, Pacific J. Math. **38**, 2 (1971), 315–323.

[10] P. CHANDRA, *Trigonometric approximation of functions in L_p-norm*, J. Math. Anal. Appl. **275**, 1 (2002), 13–26.

[11] R. A. DE VORE AND G. G. LORENTZ, *Constructive Approximation*, Springer, 1993.

[12] V. K. DZYADYK, *Introduction to the theory of uniform approximation of functions by polynomials*, "Nauka", Moscow, 1977 (in Russian).

[13] A. V. EFIMOV, *Approximation of conjugate functions by Fejér sums*, Uspekhi Mat. Nauk, , **14**, 1 (1959), 183–188.

[14] A. V. EFIMOV AND V. A. FILIMANOVA, *The approximation of conjugate functions by conjugate interepoloting sums*, Mat. Zametki, **29**, 3 (1976), 425–437 (in Russian).

[15] M. G. ESMAGANBETOV AND SHARP JACKSON, *Stechkin inequalities and the widths of classes of functions from $L_2(R^2, e^{-x^2-y^2})$*, Izv. Vyssh. Uchebn. Zaved. Mat. **51**, 2 (2007), 3–9 (in Russian); translation in Russian Math. (IZV. VUZ) **51**, 2 (2007), 1–7.

[16] M. G. ESMAGANBETOV, *Sharpening of the Jackson-Stechkin theorem on the best approximation of functions in L_p*, Izv. Akad. Nauk Kazakh, SSR Ser. Fiz. Mat. **1**, 1 (1991), 28–33 (in Russian).

[17] M. G. ESMAGANBETOV, *Minimization of exact constants in Jackson type inequalities and the widths of functions belonging to $L_2[0,2\pi]$*, Fundam, Prikl. Mat. **7**, 1 (2001), 275–280 (in Russian).

[18] B. L. GOLINSKII, *Lokal approximation of two conjugate functions by trigonometric polynomials*, Mat. Sb. **51**, 4 (1960) 401–426 (in Russian).

[19] A. GUVEN, *Trigonometric approximation of functions in weighted spaces*, Sarajevo Journal of Mathematics **5**, 17 (2009), 99–108.

[20] A. GÜVEN, D. M. ISRAFILOV, *Approximation by Means of Fourier trigonometric series in weighted Orlicz spaces*, Adv. Stud. Contemp. Math. (Kyundshang), **19**, 2 (2009), 283–295.

[21] N. A. IL'YASOV, *On an inverse theorem in the theory of approximations of periodic functions in different metric*, Mat. Zametki. **52**, 2 (1992), 53–61 (in Russian); translation in Math. Notes **52**, 1–2 (1992), 791–798.

[22] N. A. IL'YASOV, *On the direct theorem in the theory of approximations of periodic functions in different metric*, Trudy Mat. Inst. Steklova, Teor. Priblizh. Garmon. Anal., **219**, 4 (1997), 220–234 (in Russian); translation in Proc. Steklov Inst. Math. **219**, 4 (1997), 215–230.

[23] N. A. IL'YASOV, *Direct and inverse theorems in the theory of absolutely converging Fourier series on continuous periodic functions*, Izv. Ural. Gos. Univ. Mat. Mekh. **44**, 9 (2006), 89–112 (in Russian).

[24] D. M. ISRAFILOV, *Approximation p-Faber polynomials in the weighted Smirnov class and the Bieberbach polynomials*, Constr. Approx, **17**, 2 (2001), 335–351.

[25] D. M. ISRAFILOV AND R. AKGUN, *Approximation in weighted Smirnov-Orlicz classes*, J. Math. Kyoto Univ. **46**, 4 (2006), 755–770.

[26] D. M. ISRAFILOV AND A. GUVEN, *Approximation by trigonometric polynomials in weighted. Orlicz spaces*, Studia Mathematica **174**, 2 (2006), 147–168.

[27] S. Z. JAFAROV, *Approximation by rational functions in Smirnov-Orlicz classes*, Journal of Mathematical Analysis and Applications, **379**, 2 (2011), 870–877.

[28] S. Z. JAFAROV, *Approximation by polynomials and rational functions in Orlicz Spaces*, Journal of Computational Analysis and Applications, **13**, 5 (2011), 953–962.

[29] S. Z. JAFAROV, *The inverse theorem of approximation of the function in Smirnov-Orlicz classes*, Mathematical Inequalities and Applications, **120**, 4 (2012), 835–844.

[30] S. Z. JAFAROV, *On approximation in weighted Smirnov-Orlicz classes*, Complex Variables and Elliptic Equations, **570**, 5 (2012), 567–577.

[31] V. M. KOKILASHVILI, *The converse theorem of constructive theory of functions in Orlicz spaces*, Soobshch. Akad. Nauk Gruzin. SSR **37**, 2 (1965), 263–270 (in Russian).

[32] V. M. KOKILASHVILI, *On approximation of periodic functions*, Trudy Tbiliss. Mat. Inst. im. Razmadze Akad. Nauk Gruzin. SSR **340**, (1968), 51–81 (in Russian).

[33] V. M. KOKILASHVILI AND Y. E. YILDIRIR, *On the approximation in weighted Lebesgue spaces*, Proc. A. Razmadze Math. Inst. **143**, (2007), 103–113.

[34] V. M. KOKILASHVILI, *Conjugate functions*, Soobshch. Akad. Nauk Gruzin. SSR **68**, 3 (1972), 537–540.

[35] M. A. KROSNOSEL'SKII AND YA. B. RUTICKII, *Convex Functions and Orlicz Spaces*, Noordhoff, 1961.

[36] A. YU. KARLOVICH, *Algebras of singular operators with piecewise continuous coefficients on reflexive Orlicz space*, Math. Nachr. **179**, (1996), 187–222.

[37] B. MUCKENHOUPT, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165**, (1972), 207–226.

[38] A. A. PEKARSKII, *Rational approximations of absolutely continuous functions with derivative in an Orlicz space*, Math. Sb. (N.S), **117**, 1 (1982), 114–130 (in Russian).

[39] M. K. POTAPOV, *On the approximation of differentiable functions by algebraic polynomials*, Dokl. Akad. Nauk **404**, 6 (2005), 740–744.

[40] M. K. POTAPOV, *On the approximation of differentiable functions in the uniform metric*, Trudy Mat. Inst. Steklova **248**, (2005), 223-236 (in Russian); translation in Proc. Steklov Inst. Math. **248**, 1 (2005), 216–229.

[41] A.-R. K. RAMAZANOV, *On approximation by polynomials and rational functions in Orlicz Spaces*, Anal. Math. **10**, (1984), 117–132.

[42] M. M. RAO AND Z. D. REN, *Theory of Orlicz Spaces*, Maroel Dekker, New York, 1991.

[43] I. A. SHEVCHUK, *On the uniform approximation of functions on a segment*, Mat. Zametki, **40**, 1 (1986), 36–48 (in Russian).

[44] S. B. STECHKIN, *On best approximation of conjugate functions by trigonometric polynomials*, Izv. Akad. Nauk SSSR Ser. Mat. **20**, 2 (1956), 197–206 (in Russian).

[45] S. B. STECHKIN, *The approximation of periodic functions by Fejér sums*, Trudy Mat. Ins. Steklov, **62**, (1961), 522–524 (in Russian).

[46] E. S. SMAILOV AND M. G. ESMAGANBETOV, *Accuracy of estimates for the modulus of smoothness of positive order of a function in* $L_p[0,2\pi]$, $1 < p < \infty$, Functional Analysis, differential equations and their applications, Kazakh, Gos. Univ., Alma Ata, 1987, 50–54.
[47] A. F. TIMAN, *Theory of Approximation of Functions of a Real Variable*, Pergamon Press and MacMillan, 1963; Russian original published by Fizmatgiz, Moscow, 1960.
[48] A. F. TIMAN AND M. F. TIMAN, *The generalized modulus of continuity and best mean approximation*, Doklady Akad. Nauk SSSR (N. S.), **71**, (1950), 17–20.
[49] M. F. TIMAN, *Inverse theorems of the constructive theory of functions in the spaces in* L_p, Mat. Sb. **46**, 88 (1958), 125–132 (in Russian).
[50] M. F. TIMAN, *Best approximation of a function and linear methods of summing Fourier series*, Izv. Akad. Nauk SSSR Ser. Mat. **29**, 3 (1965), 587–604 (in Russian).
[51] L. V. ZHIZHIASHVILI, *Convergence of multiple trigonometric Fourier series and their conjugate trigonometric series in the spaces* L *and* C, Soobshch. Akad. Nauk Gruzin. SSR, **97**, 2 (1980), 277–279 (in Russian).
[52] L. V. ZHIZHIASHVILI, *Convergence of multiple conjugate trigonometric series in the spaces* C *and* L, Soobshch. Akad. Nauk Gruzin. SSR **125**, 2 (1987), 253–255.
[53] L. V. ZHIZHIASHVILI, *On the summation of multiple conjugate trigonometric series in the metric of the space* L_p, $p \in (0,1)$, Soobshch. Akad. Nauk Gruzin. SSR **140**, 3 (1990), 465–467.
[54] L. V. ZHIZHIASHVILI, *Trigonometric Fourier series and their conjugates*, Tbilis. Gos. Univ., Tbilisi, 1993 (in Russian); English transl.: Kluwer Acad. Publ., Dordrecht, 1996.
[55] L. V. ZHIZHIASHVILI, *On conjugate functions and Hilbert transform*, Bull. Georgian Acad. Sci. **165**, 3 (2002), 458–460.

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