Applied Mathematical Modelling 37 (2013) 3894–3902

Contents lists available at SciVerse ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Haar wavelet approximation for magnetohydrodynamic flow equations

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ARTICLE INFO

Article history:
Received 21 July 2011
Received in revised form 10 June 2012
Accepted 5 July 2012
Available online 12 September 2012

Keywords:
Haar wavelets
Partial differential equations
Approximate solution
Magnetohydrodynamic flow

ABSTRACT

This study proposes Haar wavelet (HW) approximation method for solving magnetohydrodynamic flow equations in a rectangular duct in presence of transverse external oblique magnetic field. The method is based on approximating the truncated double Haar wavelets series. Numerical solution of velocity and induced magnetic field is obtained for steady-state, fully developed, incompressible flow for a conducting fluid inside the duct. The calculations show that the accuracy of the Haar wavelet solutions is quite good even in the case of a small number of grid points. The HW approximation method may be used in a wide variety of high-order linear partial differential equations. Application of the HW approximation method showed that it is reliable, simple, fast, least computation at costs and flexible.

1. Introduction

Many approximate methods have so far been developed for solving differential equations. The usual approximate methods for partial differential equations (PDEs) are weighted residual techniques, the finite difference, the finite element and the boundary element methods [1]. Recently, various approximate methods such as spectral, pseudo-spectral, Adomian decomposition, differential transform, and Chebyshev collocation methods are discussed in the literature [2–19]. Some of the methods, previously in the literature, obtain the approximate solutions at a selected point such as finite difference and the finite element methods but, some of them such as Chebyshev collocation method use basis-functions to represent the implicit form of the approximate solutions of the problems.

Chen and Hsiao [20] derived an operational matrix of integration based on the HW method for solving ordinary differential equations (ODEs). By using the HW method, Lepik [21,22] solved higher order as well as nonlinear ODEs and some nonlinear evolution equations. Lepik [23] also used HW method to solve Burgers and sine–Gordon equations. Hariharan et al. [24,25] introduced the HW method for solving both Fisher's and FitzHugh–Nagumo equations. Çelik [26] solved Generalized Burgers–Huxley equation with HW method.

This study presents a HW method for approximately solving the linear second order PDEs with variable coefficients given in the following form:

$$A_1(x,y)\frac{\partial^2 u}{\partial x^2}+A_2(x,y)\frac{\partial^2 u}{\partial x\partial y}+A_3(x,y)\frac{\partial^2 u}{\partial y^2}+A_4(x,y)\frac{\partial u}{\partial x}+A_5(x,y)\frac{\partial u}{\partial y}+A_6(x,y)u=G(x,y) \tag{1}$$

where A_1,A_2,A_3,A_4,A_5,A_6 and G are functions of x and y defined in the interval of $a\leqslant x,y\leqslant b$. Any range $a\leqslant z,t\leqslant b$ can be transformed into the basic range $0\leqslant x,y\leqslant 1$ with the change of variables $z=(b-a)x+a$ and $t=(b-a)y+a$.

This method consists of reducing the problem to a set of algebraic equations by expanding the term that has maximum derivative given in Eq. (1) as Haar functions with unknown coefficients. The operational matrix of integration and product

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http://dx.doi.org/10.1016/j.apm.2012.07.048

operational matrix are utilized to evaluate the coefficients of the Haar functions. Identification and optimization procedures of the solutions are reduced and simplified. Since the integration of the Haar functions vector is a continuous function, the solutions obtained are continuous. In the HW approximate method, a few sparse matrixes may be obtained and there are no complex integrals or methodology. Thus, the HW method is useful for obtaining the approximate solution of PDE's, minimizing round off errors and reduction to the necessity of large computer memory. Illustrative example is given to demonstrate the application of the HW method.

2. Fundamental relations

Haar wavelet is the simplest wavelet. The Haar wavelet transform, proposed in 1909 by Alfred Haar, is the first known wavelet. Haar wavelet transform has been used as an earliest example for orthonormal wavelet transform with compact support. The Haar wavelet family for $x \in [0, 1]$ is defined as follows:

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\xi_{1,}\xi_2), \\ -1 & \text{for } x \in [\xi_2, \xi_3], \\ 0 & \text{elsewhere}, \end{cases} \tag{2}$$

where $\xi_1 = \frac{k}{m}$, $\xi_2 = \frac{k+0.5}{m}$ and $\xi_3 = \frac{k+1}{m}$. In these formulae; integer $m = 2^j$, $j = 0, 1, \ldots, J$ indicates the level of the wavelet; $k = 0, 1, \ldots, m-1$ is the translation parameter. Maximal level of resolution is J and 2^J is denoted as $M = 2^J$. The index i in Eq. (2) is calculated from the formula $i = m + k + 1$; in the case of minimal values $m = 1$, $k = 0$, we have $i = 2$. The maximal value of i is $i = 2M = 2^{J+1}$. It is assumed that the value $i = 1$ corresponds to the scaling function for which $h_1(x) = 1$ in $[0, 1]$. A set of first eight Haar functions is shown in Fig. 1, where $i = 1, 2, \ldots, 8$.

It must be noticed that all the Haar wavelets are orthogonal to each other:

$$\int_0^1 h_i(x)h_l(x)dx = \begin{cases} 2^{-j} & i = l = 2^j + k \\ 0 & i \neq l. \end{cases} \tag{3}$$

Therefore, Haar functions construct a very good transform basis. Any function $y(x)$, which is square integrable in the interval $[0, 1)$, namely $\int_0^1 y^2(x)dx$ is finite, can be expanded in a Haar series with an infinite number of terms as:

$$y(x) = \sum_{i=1}^{\infty} c_i h_i(x) \quad i = 2^j + k, j \geqslant 0, 0 \leqslant k \leqslant 2^j, x \in [0, 1), \tag{4}$$

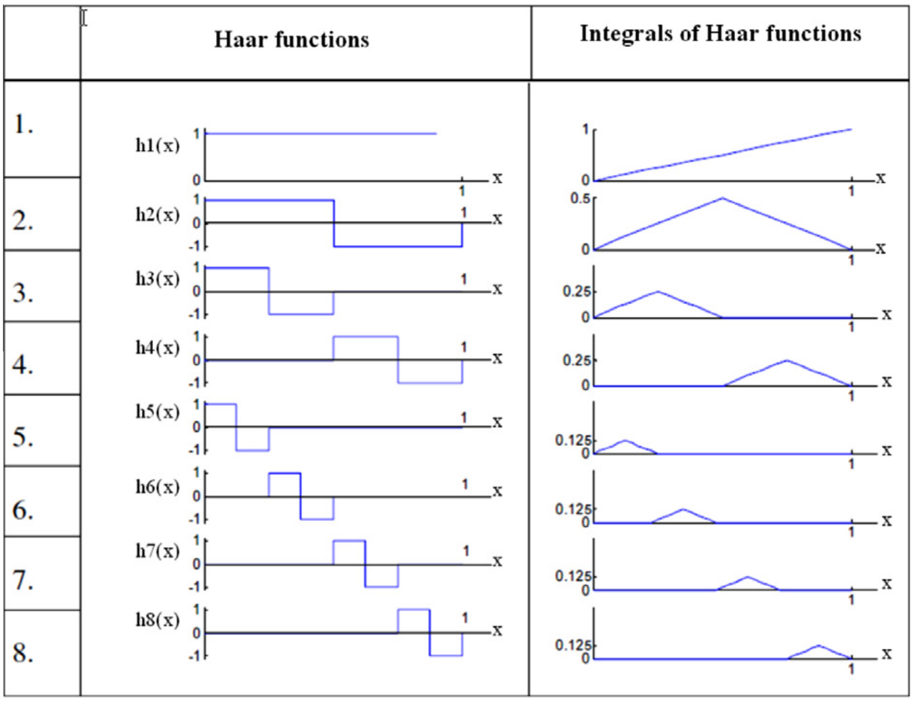

Fig. 1. First eight Haar functions and their integrals.

where the Haar coefficients,

$$c_i = 2^j \int_0^1 y(x) h_i(x) dx,$$

are determined in such a way that the integral square error

$$E = \int_0^1 \left[y(x) - \sum_{i=1}^{2M} c_i h_i(x) \right]^2 dx \tag{5}$$

is minimized.

In general, the series expansion of $y(x)$ contains infinite terms. If $y(x)$ is a piecewise constant or may be approximated as a piecewise constant during each subinterval, then $y(x)$ will be terminated at finite terms, that is

$$y(x) \cong \sum_{i=1}^{2M} c_i h_i(x) = c^T H_{2M}(x), \tag{6}$$

where the coefficient and the Haar function vectors are defined as:

$$c^T = [c_1, c_2, \dots, c_{2M}], H_{2M}(x) = [h_1(x), h_2(x), \dots, h_{2M}(x)]^T$$

respectively and $x \in [0, 1)$.

The integrals of Haar function $h_i(x)$ can be evaluated as:

$$p_{i,1}(x) = \int_0^x h_i(x) dx \tag{7}$$

$$p_{i,\upsilon}(x) = \int_0^x p_{i,\upsilon-1}(x) dx, \quad \upsilon = 2, 3, \dots. \tag{8}$$

Carrying out these integrations using Eq. (2), it is found that

$$p_{i,1}(x) = \begin{cases} x - \xi_1 & \text{for } x \in [\xi_1, \xi_2], \\ \xi_3 - x & \text{for } x \in [\xi_2, \xi_3], \\ 0 & \text{elsewhere}, \end{cases} \tag{9}$$

$$p_{i,2}(x) = \begin{cases} 0 & \text{for } x \in [0, \xi_1], \\ \frac{(x-\xi_1)^2}{2} & \text{for } x \in [\xi_1, \xi_2], \\ \frac{1}{4m^2} - \frac{(\xi_3 - x)^2}{2} & \text{for } x \in [\xi_2, \xi_3], \\ \frac{1}{4m^2} & \text{for } x \in [\xi_3, 1], \end{cases} \tag{10}$$

$$p_{i,3}(x) = \begin{cases} 0 & \text{for } x \in [0, \xi_1], \\ \frac{(x-\xi_1)^3}{6} & \text{for } x \in [\xi_1, \xi_2], \\ \frac{x-\xi_2}{4m^2} - \frac{(\xi_3 - x)^3}{6} & \text{for } x \in [\xi_2, \xi_3], \\ \frac{x-\xi_2}{4m^2} & \text{for } x \in [\xi_3, 1], \end{cases} \tag{11}$$

$$p_{i,4}(x) = \begin{cases} 0 & \text{for } x \in [0, \xi_1], \\ \frac{(x-\xi_1)^4}{24} & \text{for } x \in [\xi_1, \xi_2], \\ \frac{(x-\xi_2)^2}{8m^2} - \frac{(\xi_3 - x)^4}{24} + \frac{1}{192m^4} & \text{for } x \in [\xi_2, \xi_3], \\ \frac{(x-\xi_2)^2}{8m^2} + \frac{1}{192m^4} & \text{for } x \in [\xi_3, 1]. \end{cases} \tag{12}$$

Let us define the collocation points $x_l = (l - 0.5)/(2M),\ l = 1, 2, \dots, 2M$. By these collocation points, a discretizised form of the Haar function $h_i(x)$ can be obtained. Hence, the matrix $H(i, l) = (h_i(x_l))$, which has the dimension $2M \times 2M$, is achieved. The operational matrices of integrations $P\upsilon$, which are $2M$ square matrices, are defined by the equation $P\upsilon(i, l) = p_{i,\upsilon}(x_l)$, where υ shows the order of integration.

3. Haar wavelet method with two variable

Consider Eq. (1) with boundary conditions

$$u(0, y) = g_0(y), u(1, y) = g_1(y), u(x, 0) = f_0(x), u(x, 1) = f_1(x). \tag{13}$$

It is assumed that $u^{(2,2)}(x,y)$ can be expanded in terms of two-variable truncated Haar wavelet series as

$$u^{(2,2)}(x,y)=\sum_{r=1}^{2M}\sum_{s=1}^{2M}a_{r,s}h_r(x)h_s(y), \tag{14}$$

where $a_{r,s}$'s are Haar coefficients and $h_t(x)$ and $h_s(y)$ are Haar functions.

The series in (14) can be expressed in the matrix form as

$$u^{(2,2)}(x,y)=H_x^TAH_y,$$

where $H_x^T=[h_0(x)h_1(x)\dots h_{2M}(x)]$, $H_y=[h_0(y)h_1(y)\dots h_{2M}(y)]^T$ and

$$A=\begin{bmatrix} a_{00} & a_{01} & \cdots & a_{02M} \\ a_{10} & a_{11} & \cdots & a_{12M} \\ \vdots & \vdots & & \vdots \\ a_{2M0} & a_{2M1} & \cdots & a_{2M2M}. \end{bmatrix}$$

By integrating Eq. (14) twice with respect to x from 0 to x and twice with respect to y from 0 to y, and by using the boundary conditions, following equations are obtained

$$u^{(2,2)}(x,y)=H_x^TAH_y, \tag{15}$$

$$u^{(1,2)}(x,y)=P_1^T(r,x)AH_y+u^{(1,2)}(0,y), \tag{16}$$

$$u^{(0,2)}(x,y)=P_2^T(r,x)AH_y+xu^{(1,2)}(0,y)+g_0''(y), \tag{17}$$

$$u^{(2,1)}(x,y)=H_x^TAP_1(s,y)+u^{(2,1)}(x,0), \tag{18}$$

$$u^{(1,1)}(x,y)=P_1^T(r,x)AP_1(s,y)+u^{(1,1)}(x,0)-u^{(1,1)}(0,0)+u^{(1,1)}(0,y), \tag{19}$$

$$u^{(0,1)}(x,y)=P_2^T(r,x)AP_1(s,y)+u^{(0,1)}(x,0)-xu^{(1,1)}(0,0)+xu^{(1,1)}(0,y)+g_0'(y)-g_0'(0), \tag{20}$$

$$u^{(2,0)}(x,y)=H_x^TAP_2(s,y)+yu^{(2,1)}(x,0)+f_0''(x), \tag{21}$$

$$u^{(1,0)}(x,y)=P_1^T(r,x)AP_2(s,y)+u^{(1,0)}(0,y)-yu^{(1,1)}(0,0)+yu^{(1,1)}(x,0)+f_0'(x)-f_0'(0), \tag{22}$$

$$u^{(0,0)}(x,y)=P_2^T(r,x)AP_2(s,y)+x[u^{(1,0)}(0,y)-f_0'(x)]+y[u^{(1,1)}(x,0)-g_0'(0)-xu^{(1,1)}(0,0)]+f_0(x)-f_0(0)+g_0(y), \tag{23}$$

where $P_1^T(r,x)=\int_0^x H_x^T dx$, $P_x^T(r,x)=\int_0^x\int_0^x H_x^T dxdx$, $P_1(s,y)=\int_0^y H_y dy$ and $P_2(s,y)=\int_0^y\int_0^y H_y dydy$.

Putting $x = 1$ and $y = 1$in formulae (23) respectively, the following formulas can be obtained.

$$u^{(1,1)}(0,0)=P_2^T(r,1)AP_2(s,1)+g_0(1)-g_0'(0)+g_1'(0)+f_0(1)-f_0(0)-f_0'(0)+f_1'(0)-f_1(1), \tag{24}$$

$$\begin{aligned} u^{(1,0)}(0,y)={}&y[P_{2_2}^T(r,1)AP_2(s,1)+g_0(1)+f_0(1)-f_0(0)-f_0'(0)+f_1'(0)-f_1(1)]-P_2^T(r,1)AP_2(s,y)-g_0(y)+g_1(y)\\ &-f_0(1)+f_0(0)+f_0'(0), \end{aligned} \tag{25}$$

$$\begin{aligned} u^{(0,1)}(x,0)={}&x[P_2^T(r,1)AP_2(s,1)+g_0(1)-g_0'(0)+g_1'(0)+f_0(1)-f_0(0)-f_1(1)]-P_2^T(r,x)AP_2(s,1)-g_0(y)+g_1(y)\\ &-f_0(1)+f_0(0)+f_0'(0). \end{aligned} \tag{26}$$

Putting $x = 1$ in formulae (20) and $y = 1$ in formulae (22), the formulas given as:

$$u^{(1,1)}(0,y)=P_2^T(r,1)AP_2(s,1)-P_2^T(r,1)AP_1(s,y)-g_0'(y)+g_1'(y)+f_0(1)-f_0(0)-f_0'(0)-f_1(1)+f_1'(0), \tag{27}$$

$$u^{(1,1)}(x,0)=P_2^T(r,1)AP_2(s,1)-P_1^T(r,x)AP_2(s,1)+g_0(1)-g_0'(0)+g_1'(0)-g_1(1)-f_0'(x)+f_0(1)-f_0(0)+f_1'(x) \tag{28}$$

can be obtained respectively.

Also putting $x = 1$ in formulae (17) and $y = 1$ in formulae (21), we have the following expressions are obtained respectively.

$$u^{(1,2)}(0,y)=-P_2^T(r,1)AH_y-g_0''(y)+g_1''(y), \tag{29}$$

$$u^{(2,1)}(x,0)=-H_x^TAP_2(s,1)-f_0''(x)+f_1''(x). \tag{30}$$

When substituting Eqs. (17), (19), (20)–(30) into Eq. (1), equation including matrix representation like as $P_2^T(r,1)AH_y$, $P_2^T(r,1)AP_2(s,1)$, $P_2^T(r,1)AP_1(s,y), \ldots$ etc. can be obtained.

Matrix equation like as WAY can be transformed into a new matrix equation XC by using

$$x_{1,(j-1)N+k} = w_{1,j}y_{k,1}, \quad j,k = 1,2,\ldots,2M, \tag{31}$$

where

$$C = [a_{00}a_{01}\cdots a_{02M}a_{10}a_{11}\cdots a_{12M}\cdots a_{2M0}a_{2M1}\cdots a_{2M2M}]^T, \quad X = \lfloor x_{1,q} \rfloor \; q = 1,2,3,\ldots,2M.$$

Thus, by summing matrixes in the form XC, Eq. (1) can be written as:

$$\tilde{W}C = G(x,y), \tag{32}$$

where $\tilde{W} = [\tilde{w}_{1,q}],\; q = 1,2,3,\ldots,2M$ and $\tilde{w}_{i,q}$ is also a function of x and y.

Let us define the collocation points

$$x_l = (l-0.5)/(2M), \quad y_l = (l-0.5)/(2M) \quad l = 1,2,\ldots,2M.$$

By substituting the collocation points in Eq. (32), the following algebraic equation systems can be constructed as

$$UC = G(x_i,y_j) \quad i,j = 1,2,\ldots,2M, \tag{33}$$

where U is a $2M \times 2M$ dimensional matrix.

By solving algebraic equation systems (33) subject to boundary conditions (13), we can find the coefficients of the Haar wavelet series. Substituting Haar wavelet coefficients into Eq. (23), we have the approximate solution of the PDE (1) with boundary conditions (13).

4. Application of the HW method to magnetohydrodynamic flow problem

Basic equations of fluid mechanics and Maxwell equations of electromagnetism are well known as coupled system of equations for velocity and magnetic field. In a rectangular duct $\Omega \subset R^2$, for the equations of steady, laminar, fully developed flow of viscous, incompressible and electrically conducting fluid are subjected to a constant and uniform applied magnetic field. They can be put in non-dimensional form [27] as

$$\begin{array}{l} \nabla^2 V + H_{ax^*}\frac{\partial B}{\partial x^*} + H_{ay^*}\frac{\partial B}{\partial y^*} = -1 \\ \nabla^2 B + H_{ax^*}\frac{\partial V}{\partial x^*} + H_{ay^*}\frac{\partial V}{\partial y^*} = 0 \end{array} \quad \text{in}\Omega \tag{34}$$

with the boundary conditions

$$V = B = 0 \quad \text{on} \quad \partial\Omega. \tag{35}$$

$V(x^*,y^*)$ and $B(x^*,y^*)$ in Eq. (34) are velocity and induced magnetic field, respectively. The boundaries of the duct are assumed to be insulated. Hartmann number H_a is the norm of the vector $\mathrm{H_a} = (H_{ax^*}, H_{ay^*})$.

The fluid is driven down the duct by means of a constant pressure gradient and $V(x^*,y^*)$, $B(x^*,y^*)$ are parallel to z^* axis which is the axis of the duct. When the applied magnetic field intensity B_0 acts in a direction lying in the x^*y^* plane but forming an angle α with the y-axis, the following can be obtained:

$$H_{ax^*} = H_a \sin\alpha, \quad H_{ay^*} = H_a \cos\alpha, \quad H_a = \left(H_{ax^*}^2 + H_{ay^*}^2\right)^{\frac{1}{2}}.$$

With two new variables $U_1 = V + B,\; U_2 = V - B$, Eq. (34) can be transformed into the following set of equations:

$$\begin{array}{l} \nabla^2 U_1 + H_{ax^*}\frac{\partial U_1}{\partial x^*} + H_{ay^*}\frac{\partial U_1}{\partial y^*} = -1 \\ \nabla^2 U_2 - H_{ax^*}\frac{\partial U_2}{\partial x^*} - H_{ay^*}\frac{\partial U_2}{\partial y^*} = -1 \end{array} \quad \text{in} -1 \leqslant x^*, y^* \leqslant 1 \tag{36}$$

with the boundary conditions $U_1 = U_2 = 0$ on $\partial\Omega$. If U_1 is solved as $U_1(H_{ax^*}, H_{ay^*})$ from Eq. (36), then $U_2 = U_1(-H_{ax^*}, -H_{ay^*})$ can directly be obtained. Solution of Eq. (34) can be obtained by substituting $V = \frac{U_1+U_2}{2},\; B = \frac{U_1-U_2}{2}$.

By transforming the range $-1 \leqslant x^*, y^* \leqslant 1$ into the basic range $0 \leqslant x, y \leqslant 1$ then Eq. (36) can be written as

$$\begin{array}{l} \nabla^2 U_1 + 2H_{ax}\frac{\partial U_1}{\partial x} + 2H_{ay}\frac{\partial U_1}{\partial y} = -4 \\ \nabla^2 U_2 - 2H_{ax}\frac{\partial U_2}{\partial x} - 2H_{ay}\frac{\partial U_2}{\partial y} = -4 \end{array} \quad \text{in } 0 < x, y < 1 \tag{37}$$

with the boundary conditions $U_1(x,0) = 0,\; U_1(x,1) = 0\;\; U_1(0,y) = 0,\; U_1(1,y) = 0$ and $U_2(x,0) = 0,\; U_2(x,1) = 0\;\; U_2(0,y) = 0,$ $U_2(1,y) = 0$

It is assumed that $u^{(2,2)}(x,y)$, maximum derivative term of Eq. (37), can be expanded in terms of two-variable truncated Haar wavelets series as

$$u^{(2,2)}(x,y)=\sum_{r=1}^{2M}\sum_{s=1}^{2M}a_{r,s}h_r(x)h_s(y).$$

By integrating Haar wavelets series twice with respect to x from 0 to x and twice with respect to y from 0 to y, and by using the boundary conditions following equations are obtained

$$u^{(2,2)}(x,y)=H_x^TAH_y,$$

$$u^{(1,2)}(x,y)=P_1^T(r,x)AH_y+u^{(1,2)}(0,y),$$

$$u^{(0,2)}(x,y)=P_2^T(r,x)AH_y+xu^{(1,2)}(0,y),$$

$$u^{(2,1)}(x,y)=H_x^TAP_1(s,y)+u^{(2,1)}(x,0),$$

$$u^{(1,1)}(x,y)=P_1^T(r,x)AP_1(s,y)+u^{(1,1)}(x,0)-u^{(1,1)}(0,0)+u^{(1,1)}(0,y),$$

$$u^{(0,1)}(x,y)=P_2^T(r,x)AP_1(s,y)+u^{(0,1)}(x,0)-xu^{(1,1)}(0,0)+xu^{(1,1)}(0,y),$$

$$u^{(2,0)}(x,y)=H_x^TAP_2(s,y)+yu^{(2,1)}(x,0),$$

$$u^{(1,0)}(x,y)=P_1^T(r,x)AP_2(s,y)+u^{(1,0)}(0,y)-yu^{(1,1)}(0,0)+yu^{(1,1)}(x,0),$$

$$u^{(0,0)}(x,y)=P_2^T(r,x)AP_2(s,y)+xu^{(1,0)}(0,y)+y[u^{(1,1)}(x,0)-xu^{(1,1)}(0,0)],$$

$$u^{(1,1)}(0,0)=P_2^T(r,1)AP_2(s,1),$$

$$u^{(1,0)}(0,y)=yP_2^T(r,1)AP_2(s,1)-P_2^T(r,1)AP_2(s,y),$$

$$u^{(0,1)}(x,0)=xP_2^T(r,1)AP_2(s,1)\ -P_2^T(r,x)AP_2(s,1),$$

$$u^{(1,1)}(0,y)=P_2^T(r,1)AP_2(s,1)-P_2^T(r,1)AP_1(s,y),$$

$$u^{(1,1)}(x,0)=P_2^T(r,1)AP_2(s,1)-P_1^T(r,x)AP_2(s,1),$$

$$u^{(1,2)}(0,y)=-P_2^T(r,1)AH_y,$$

$$u^{(2,1)}(x,0)=-H_x^TAP_2(s,1).$$

By substituting these equations in the first Equation of (37), the following equation can be obtained:

$$\begin{aligned}&H_x^TAP_2(s,y)-yH_x^TAP_2(s,1)+P_2^T(r,x)AH_y-xP_2^T(r,1)AH_y\\&\quad+2H_{ax}[P_1^T(r,x)AP_2(s,y)+yP_2^T(r,1)AP_2(s,1)-P_2^T(r,1)AP_2(s,y)-yP_1^T(r,x)AP_2(s,1)]\\&\quad+2H_{ay}[P_2^T(r,x)AP_1(s,y)-P_2^T(r,x)AP_2(s,1)+xP_2^1(r,1)AP_2(s,1)-xP_2^T(r,1)AP_1(s,y)]=-4.\end{aligned}\tag{38}$$

Matrix equation like as WAY can be transformed into a new matrix equation XC by using

$$x_{1,(j-1)N+k}=w_{1,j}y_{k,1},\quad j,k=1,2,\ldots,2M.$$

By summing matrixes in the form XC, Eq. (32) can be obtained

By substituting the collocation points, defined as

$$x_l=(l-0.5)/(2M),\quad y_l=(l-0.5)/(2M)\quad l=1,2,\ldots,2M$$

into Eq. (32), the algebraic equation systems Eq. (33) can be constructed. If we solve the equation systems, we can find the coefficients of the Haar wavelet series. By substituting Haar wavelet coefficients into the equation

$$u^{(0,0)}(x,y)=P_2^T(r,x)AP_2(s,y)+xu^{(1,0)}(0,y)+y[u^{(1,1)}(x,0)-xu^{(1,1)}(0,0)],$$

we have the approximate solution of the first equation of (37) with the boundary conditions $U_1(x,0)=0$, $U_1(x,1)=0$ $U_1(0,y)=0$, $U_1(1,y)=0$.

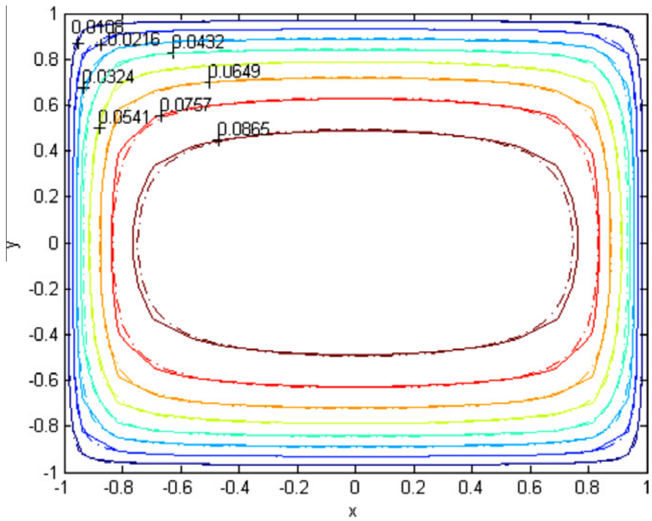

Fig. 2. Fig. 1 Velocity, $H_a = 10$, $M = 8$.

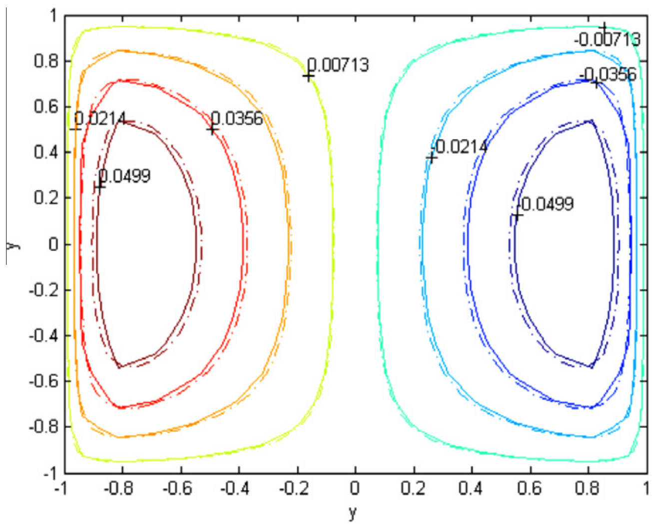

Fig. 3. Magnetic field, $H_a = 10$, $M = 8$.

5. Numerical results

The Haar wavelet method is applied to solve the equation for $\alpha = \pi/2$

$$\nabla^2 U_1 + 2H_a \frac{\partial U_1}{\partial x} = -4 \text{in } 0 < x, y < 1$$

with the boundary conditions $U_1(x, 0) = 0$, $U_1(x, 1) = 0$, $U_1(0, y) = 0$, $U_1(1, y) = 0$, $U_1 = 0$ on $\partial\Omega$. Solution U_2 can be obtained from solution U_1 as $U_2 = U_1(-H_a)$, which also satisfies $U_2 = 0$ on $\partial\Omega$.

Solution of resulting algebraic linear system of equations, which was constructed by applying the Haar wavelet method in Eq. (38), was obtained by the MATLAB. The solutions were obtained for values of M and Hartmann number H_a for the case that the applied magnetic field is parallel to the x axis ($\alpha = \pi/2$). Solutions of the Haar wavelet method which are shown by solid line (—) were compared with Shercliff's [28] exact solutions for $\alpha = \pi/2$ which are shown by dash-dotted line (– –) in figures.

Figs. 2 and 3 present velocity and induced magnetic field contours comparing with the exact solution for $H_a = 10$ and $M = 8$. Similarly, Figs. 4 and 5 show the results for $H_a = 50$ and $M = 32$. As can be seen in Figs. 2–5, the computed and actual values are overlapped. One can notice that when H_a is increased, the velocity and induced magnetic field become uniform at

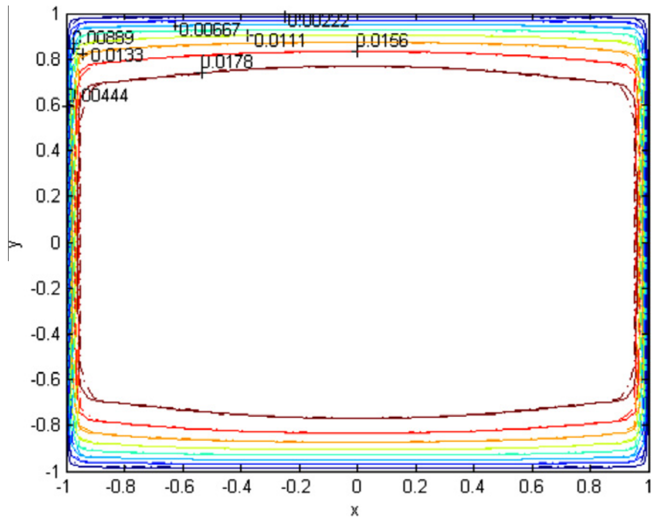

Fig. 4. Velocity, $H_a = 50$, $M = 32$.

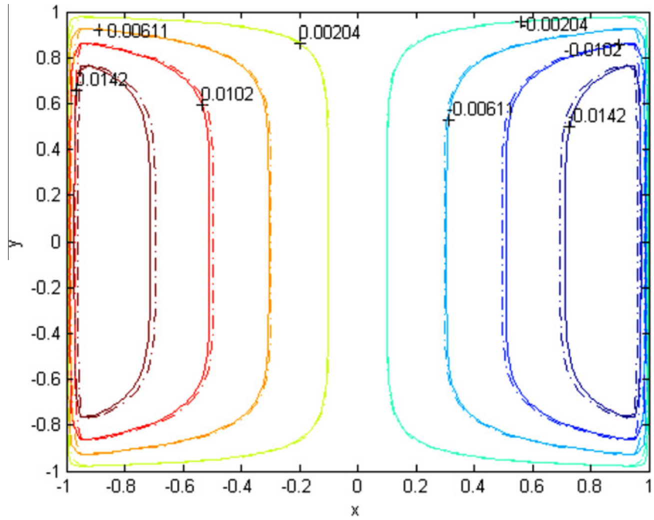

Fig. 5. Magnetic field, $H_a = 50$, $M = 32$.

the center of the duct and flow becomes stagnant. The boundary layer formation close to the walls for both velocity and induced magnetic field is well observed for high Hartmann number. These are the well-known characteristics of magnetohydrodynamic flow and are in agreement with our results.

6. Conclusion

This paper proposes two-dimensional Haar wavelet approach for the magnetohydrodynamic flow equations. Approximate solutions of the magnetohydrodynamic flow equations in a rectangular duct in the presence of transverse external oblique magnetic field, obtained by computer simulation, are compared with exact solutions. These calculations demonstrate that the accuracy of the Haar wavelet solutions is quite high even in the case of a small number of grid points. In proposed HW method, there are no complex integrals or methodology except a few construction of spars transform matrix. Applications of HW method are quite simple and it also gives the implicit form of the approximate solutions of the problems. These are the main advantages of the HW method except exponential increase of computer memory. This method is also very convenient for solving the boundary value problems, since the boundary conditions in the solution are automatically taken into account.

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