Mathematical and Computer Modelling 55 (2012) 1763–1769

Contents lists available at SciVerse ScienceDirect

Mathematical and Computer Modelling

journal homepage: www.elsevier.com/locate/mcm

Matrix operators on sequence A_k

Mehmet Ali Sarigöl

Department of Mathematics, University of Pamukkale, Denizli 20007, Turkey

ARTICLE INFO

Article history:
Received 1 March 2011
Received in revised form 5 November 2011
Accepted 7 November 2011

Keywords:
Bounded operator
Banach sequence space
Absolute summability

ABSTRACT

This paper gives necessary or sufficient conditions for a triangular matrix T to be a bounded operator from A_k to A_r, i.e., $T \in B(A_k, A_r)$ for the case $r \geq k \geq 1$ where A_k is defined by (1.3), and so extends some well-known results.

1. Introduction

Let $\sum x_v$ be a given infinite series with partial sums (s_n), and let T be an infinite matrix with complex numbers. By $(T_n(s))$ we denote the T-transform of the sequence $s = (s_n)$, i.e.,

$$T_n(s) = \sum_{v=0}^{\infty} t_{nv} s_v, \quad n, v = 0, 1, 2, \ldots. \tag{1.1}$$

The series $\sum a_v$ is then said to be k-absolutely summable by T for $k \geq 1$, written by $|T|_k$, if

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta T_{n-1}(s)|^k < \infty \tag{1.2}$$

where Δ is the forward difference operator defined by $\Delta T_{n-1}(s) = T_{n-1}(s) - T_n(s)$, [1].

A matrix T is said to be a bounded linear operator from A_k to A_r, denoted by $T \in B(A_k, A_r)$, if $T : A_k \to A_r$, where

$$A_k = \left\{ (S_v) : \sum_{v=1}^{\infty} v^{k-1} |\Delta S_{v-1}|^k < \infty \right\}. \tag{1.3}$$

In 1970, Das [2] defined a matrix T to be absolutely k-th power conservative for $k \geq 1$, denoted by $B(A_k)$, i.e., if $(T_n(s)) \in A_k$ for every sequence $(s_n) \in A_k$, and also proved that every conservative Hausdorff matrix $H \in B(A_k, A_k)$, i.e., $H \in B(A_k)$.

Let (C, α) denote the Cesáro matrix of order $\alpha > -1$, σ_n^α its n-th transform of a sequence (s_n). Using $T_n(s) = \sigma_n^\alpha$, Flett [3] proved that, if a series $\sum x_n$ is summable $|C, \alpha|_k$, then it is also summable $|C, \beta|_r$ for each $r \geq k > 1$ and $\beta \geq \alpha + 1/k - 1/r$, or $r \geq k \geq 1$ and $\beta > \alpha + 1/k - 1/r$. Setting $\alpha = 0$ gives an inclusion type theorem for Cesáro matrices.

Recently, Savaş and Şevli [4] have proved the following theorem dealing with an extension of Flett's result. Some authors have also attributed to generalize the result of Flett. For example [5], see.

E-mail addresses: msarigol@pau.edu.tr, msarigol@pamukkale.edu.tr.

0895-7177/$ – see front matter
doi:10.1016/j.mcm.2011.11.024

Theorem 1.1. *Let $r \geq k \geq 1$.*

(i) *It holds $(C, \alpha) \in B(A_k, A_r)$ for each $\alpha > 1 - k/r$.*
(ii) *If $\alpha = 1 - k/r$ and the condition $\sum_{n=1}^{\infty} n^{k-1} \log n |a_n|^k = O(1)$ is satisfied, then $(C, \alpha) \in B(A_k, A_r)$.*
(iii) *If the condition $\sum_{n=1}^{\infty} n^{k+(r/k)(1-\alpha)-2} |a_n|^k = O(1)$ is satisfied then $(C, \alpha) \in B(A_k, A_r)$ for each $-k/r < \alpha < 1 - k/r$.*

It should be noted that Part (i) of Theorem 1.1 is easily obtained from Flett's result since $\alpha > 1 - k/r = (r-k)/r \geq (r-k)/rk = 1/k - 1/r$ for $r \geq k \geq 1$. Also, Parts (ii) and (iii) are not correct. In fact, if $-k/r < \alpha < 1 - k/r$, then $(r/k)(1-\alpha) - 1 > 0$ and so

$$\sum_{n=1}^{\infty} n^{k-1} |a_n|^k \leq \sum_{n=1}^{\infty} n^{k+(r/k)(1-\alpha)-2} |a_n|^k < \infty$$

and also

$$\sum_{n=1}^{\infty} n^{k-1} |a_n|^k \leq \sum_{n=1}^{\infty} \log n n^{k-1} |a_n|^k < \infty.$$

This means that (C, α) maps a proper subset of A_k to A_r, and hence $(C, \alpha) \notin B(A_k, A_r)$.

Motivated by Theorem 1.1, a natural problem is what the sufficient conditions are for $T \in B(A_k, A_r)$, where T is any lower triangular matrix and $k, r \geq 1$.

2. Main results

The aim of this paper is to answer the above problem for $r \geq k \geq 1$ by establishing the following theorems which give us more than we need, and also deduce various known results.

Given a lower triangular matrix $T = (t_{nv})$, we can associate with T two matrices $\overline{T} = (\bar{t}_{nv})$ and $\widehat{T} = (\widehat{t}_{nv})$ defined by

$$\bar{t}_{nv} = \sum_{j=v}^{n} t_{nj}, \quad n, v = 0, 1, \ldots, \widehat{t}_{00} = \bar{t}_{00} = t_{00}, \qquad \widehat{t}_{nv} = \bar{t}_{nv} - \bar{t}_{n-1,v}, \quad n = 1, 2, \ldots.$$

Then

$$T_n(s) = \sum_{v=0}^{n} t_{nv} s_v = \sum_{v=0}^{n} t_{nv} \sum_{i=0}^{v} x_i = \sum_{i=0}^{n} x_i \sum_{v=i}^{n} t_{nv} = \sum_{v=0}^{n} \bar{t}_{nv} x_v$$

and

$$\Delta T_{n-1}(s) = \sum_{v=0}^{n} \bar{t}_{nv} x_v - \sum_{v=0}^{n-1} \bar{t}_{n-1,v} x_v = -\sum_{v=0}^{n} \widehat{t}_{nv} x_v, \quad (\bar{t}_{n-1,n} = 0). \tag{2.1}$$

Thus, $B(A_k, A_r)$ means that

$$\sum_{n=1}^{\infty} n^{k-1} |x_n|^k < \infty \Rightarrow \sum_{n=1}^{\infty} n^{r-1} |\Delta T_{n-1}(s)|^r < \infty.$$

With these notations we have the following.

Theorem 2.1. *Let $T = (t_{nv})$ be a lower triangular matrix. Then $T \in B(A_k, A_r)$ for $r \geq k \geq 1$ if*

$$\sum_{n=v}^{\infty} n^{r-1} d_n^{r/k'} |\widehat{t}_{nv}|^{r/\mu} = O(1) \tag{2.2}$$

and

$$\sum_{n=1}^{\infty} n^{r-1} |\widehat{t}_{n0}|^r < \infty, \tag{2.3}$$

where $\mu = 1 + r/k'$,

$$d_n = \sum_{v=1}^{n} v^{-1} |\widehat{t}_{nv}|^{k'/\mu'},$$

k' and μ' are the conjugates of k and μ.

Proof. Let $r \geq k \geq 1$. By applying Hölder's inequality in (2.1) we have

$$\begin{aligned} |\Delta T_{n-1}(s)| &\leq \sum_{v=0}^{n} |\widehat{t}_{nv}|\,|x_v| \\ &= |\widehat{t}_{n0}|\,|x_0| + \sum_{v=1}^{n} \left(v^{1/k'}|\widehat{t}_{nv}|^{1/\mu}|x_v|\right)\left(v^{-1/k'}|\widehat{t}_{nv}|^{1/\mu'}\right) \\ &\leq |\widehat{t}_{n0}|\,|x_0| + \left(\sum_{v=1}^{n} |\widehat{t}_{nv}|^{k/\mu} v^{k-1}|x_v|^k\right)^{1/k}\left(\sum_{v=1}^{n} v^{-1}|\widehat{t}_{nv}|^{k'/\mu'}\right)^{1/k'}. \end{aligned} \tag{2.4}$$

The last factor on the right of (2.4) is to be omitted if $k = 1$. Further, since $(x_v) \in A_k$, it follows from Hölder's inequality with indices $r/k,\ r/(r-k)$ that

$$\begin{aligned} \sum_{v=1}^{n} |\widehat{t}_{nv}|^{k/\mu} v^{k-1}|x_v|^k &= \sum_{v=1}^{n}\left\{|\widehat{t}_{nv}|^{k/\mu} v^{\frac{-k}{r}+\frac{k^2}{r}}|x_v|^{\frac{k^2}{r}}\right\}\left\{v^{\frac{-(r-k)+k(r-k)}{r}}|x_v|^{\frac{k(r-k)}{r}}\right\} \\ &\leq \left(\sum_{v=1}^{n} |\widehat{t}_{nv}|^{r/\mu} v^{k-1}|x_v|^k\right)^{k/r}\left(\sum_{v=1}^{n} v^{k-1}|x_v|^k\right)^{(r-k)/r} \\ &= O(1)\left(\sum_{v=1}^{n} |\widehat{t}_{nv}|^{r/\mu} v^{k-1}|x_v|^k\right)^{k/r}, \end{aligned} \tag{2.5}$$

which implies that

$$|\Delta T_{n-1}(s)|^r = O(1)\left(|\widehat{t}_{n0}|^r + d_n^{r/k'}\sum_{v=1}^{n} |\widehat{t}_{nv}|^{r/\mu} v^{k-1}|x_v|^k\right).$$

The second factor of (2.5) is to be omitted if $r = k$. Therefore by (2.2) and (2.3) we get

$$\begin{aligned} \sum_{n=1}^{\infty} n^{r-1}|\Delta T_{n-1}(s)|^r &= O(1)\left(\sum_{n=1}^{\infty} n^{r-1}|\widehat{t}_{n0}|^r + \sum_{n=1}^{\infty} n^{r-1}d_n^{r/k'}\sum_{v=1}^{n} |\widehat{t}_{nv}|^{r/\mu} v^{k-1}|x_v|^k\right) \\ &= O(1)\left(\sum_{n=1}^{\infty} n^{r-1}|\widehat{t}_{n0}|^r + \sum_{v=1}^{\infty} v^{k-1}|x_v|^k \sum_{n=v}^{\infty} n^{r-1}d_n^{r/k'}|\widehat{t}_{nv}|^{r/\mu}\right) < \infty \\ &= O(1)\left(\sum_{n=1}^{\infty} n^{r-1}|\widehat{t}_{n0}|^r + \sum_{v=1}^{\infty} v^{k-1}|x_v|^k\right) < \infty, \end{aligned}$$

which completes the proof. □

The following theorem establishes the necessary conditions for $T \in B(A_k, A_r)$.

Theorem 2.2. *Let $T = (t_{nv})$ be a lower triangular matrix. If $T \in B(A_k, A_r)$ for $r \geq k \geq 1$, then*

$$\sum_{n=v}^{\infty} n^{r-1}|\widehat{t}_{nv}|^r = O(v^{r/k'}) \quad \text{as } v \to \infty. \tag{2.6}$$

Proof. It is routine to verify that A_k is a Banach space and also K-space (i.e., the coordinate functionals are continuous) if normed by

$$\|s\| = \left(|s_0|^k + \sum_{v=1}^{\infty} v^{k-1}|\Delta S_{v-1}|^k\right)^{1/k} = \left(|x_0|^k + \sum_{v=1}^{\infty} v^{k-1}|\,x_v|^k\right)^{1/k}.$$

Hence the map $T : A_k \to A_r$ is continuous, i.e., there exists a constant $M > 0$ such that $\|T(x)\| \leq M\|x\|$, equivalently

$$\left(|T_0(s)|^r + \sum_{n=1}^{\infty} n^{r-1}\left|\sum_{v=0}^{n}\widehat{t}_{nv}x_v\right|^r\right)^{1/r} \leq M\left(|x_0|^k + \sum_{v=1}^{\infty} v^{k-1}|x_v|^k\right)^{1/k} \tag{2.7}$$

for all $x \in A_k$. Taking any $v \geq 1$, if we apply (2.1) with $x_v = 1$, $x_n = 0 (n \neq v)$, then we obtain

$$\Delta T_{n-1}(s) = \begin{Bmatrix} 0, & \text{if } n < v \\ -\widehat{t}_{nv}, & \text{if } n \geq v \end{Bmatrix},$$

and so

$$\left(\sum_{n=v}^{\infty} n^{r-1} |\widehat{t}_{nv}|^r \right)^{1/r} \leq M (v^{k-1})^{1/k}$$

by (2.7), which is equivalent to (2.6). □

Corollary 2.3. *Let $T = (t_{nv})$ be a lower triangular matrix. Then $T \in B(A_1, A_r)$ for $r \geq 1$ if and only if*

$$\sum_{n=v}^{\infty} n^{r-1} |\widehat{t}_{nv}|^r = O(1) \quad \text{as } v \to \infty. \tag{2.8}$$

In order to justify the fact that results of Theorems 2.1 and 2.2 are significant, we give some applications.

Lemma 2.4 ([6]). *Let $1 \leq k < \infty$, $\beta > -1$ and $\sigma < \beta$. For $v \geq 1$, let $E_v = \sum_{n=v}^{\infty} \frac{|A_{n-v}^{\sigma}|^k}{n (A_n^{\beta})^k}$. Then, if $k = 1$,*

$$E_v = \begin{Bmatrix} O(v^{-\beta-1}), & \text{if } \sigma \leq -1 \\ O(v^{-\beta+\sigma}), & \text{if } \sigma > -1 \end{Bmatrix}.$$

If $1 < k < \infty$, then

$$E_v = \begin{Bmatrix} O(v^{-k\beta-1}), & \text{if } \sigma < -1/k \\ O(v^{-k\beta-1} \log v), & \text{if } \sigma = -1/k \\ O(v^{-k\beta+k\sigma}), & \text{if } \sigma > -1/k \end{Bmatrix}$$

we apply Theorems 2.1 and 2.2 to the Cesáro matrix of order $\alpha > -1$ in which the matrix T is given by $t_{nv} = (A_{n-v}^{\alpha-1})/A_n^{\alpha}$. It is well-known that (see [7]) $\overline{t}_{nv} = A_{n-v}^{\alpha}/A_n^{\alpha}$ and $\widehat{t}_{nv} = v A_{n-v}^{\alpha-1}/(n A_n^{\alpha})$.

Thus, considering Lemma 2.4 and Theorem 2.1, we get the following result of Flett.

Corollary 2.5. (i) *If $r \geq k \geq 1$ and $\alpha > 1/k - 1/r$, then $(C, \alpha) \in B(A_k, A_r)$.*
(ii) *If $r \geq k \geq 1$ and $-1 < \alpha < 1/k - 1/r$, then $(C, \alpha) \notin B(A_k, A_r)$.*
(iii) *If $r = k \geq 1$ and $\alpha = 1/k - 1/r$, then $(C, \alpha) \in B(A_k, A_r)$.*

Proof. (i) Let $\alpha > 1/k - 1/r$. If $r \geq k > 1$, then it is seen that $\mu/r = \mu'/k' = 1 - 1/k + 1/r$ and $r(\alpha - 1)/\mu = k'(\alpha - 1)/\mu' > -1$. Thus it follows that

$$d_n = \sum_{v=1}^{n} v^{-1} \left| \frac{v A_{n-v}^{\alpha-1}}{n A_n^{\alpha}} \right|^{k'/\mu'} = O(n^{-k'/\mu'}),$$

and so

$$E_v = \sum_{n=v}^{\infty} n^{r-1} (d_n)^{r/k'} |\widehat{t}_{nv}|^{r/\mu} = O(1) \sum_{n=v}^{\infty} n^{r-1} n^{-r/\mu'} \left| \frac{v A_{n-v}^{\alpha-1}}{n A_n^{\alpha}} \right|^{r/\mu}$$
$$= O(v^{r/\mu}) \sum_{n=v}^{\infty} \frac{|A_{n-v}^{\alpha-1}|^{r/\mu}}{n (A_n^{\alpha})^{r/\mu}} = O(1) \quad \text{as } v \to \infty,$$

by Lemma 2.4. Hence, the proof of (i) is completed by Theorem 2.1.

(ii) If $-1 < \alpha < 1/k - 1/r$, then $v^{(1/k)-(1/r)} t_{vv} = v^{(1/k)-(1/r)} \frac{A_0^{\alpha-1}}{A_v^{\alpha}} \cong v^{(1/k)-(1/r)-\alpha} \neq O(1)$, i.e., the condition (2.6) is not satisfied, and so the result is seen from Theorem 2.2.

(iii) is clear from Lemma 2.4 and Theorem 2.1. □

A discrete generalized Cesáro matrix (see [8]) is a triangular matrix T with nonzero entries $t_{nv} = \lambda^{n-v}/(n+1)$, where $0 \leq \lambda \leq 1$.

Corollary 2.6. *Let C_{λ} be a Rhaly discrete matrix. Then $C_{\lambda} \in B(A_k, A_r)$ for $0 < \lambda < 1$ and $r \geq k \geq 1$.*

Proof. In Theorem 2.1, take $T = C_\lambda$. If $r \geq k > 1$, then $k'/\mu' \geq 1$. Now we have

$$|\widehat{t}_{nv}| = \left| \frac{1}{n+1} \sum_{j=0}^{n} \lambda^{n-j} - \frac{1}{n} \sum_{j=0}^{n-1} \lambda^{n-1-j} - \frac{\lambda^n}{n+1} \sum_{j=0}^{v-1} \left(\frac{1}{\lambda}\right)^j + \frac{\lambda^{n-1}}{n} \sum_{j=0}^{v-1} \left(\frac{1}{\lambda}\right)^j \right|$$
$$= \left| \frac{1}{n+1} \left(\frac{1-\lambda^{n+1}}{1-\lambda} \right) - \frac{1}{n} \left(\frac{1-\lambda^n}{1-\lambda} \right) + \left(\frac{\lambda^{n-1}}{n} - \frac{\lambda^n}{n+1} \right) \frac{\lambda}{1-\lambda} \left(\left(\frac{1}{\lambda} \right)^v \right) - 1 \right|$$
$$= O(1) \left(\frac{1}{n^2} + \frac{\lambda^{n-v}}{n} \right)$$

and so

$$d_n = \sum_{v=1}^{n} \frac{1}{v} |\widehat{t}_{nv}|^{k'/\mu'} = O(1) \left(\frac{\log n}{n^{2k'/\mu'}} + \frac{1}{n^{k'/\mu'}} \sum_{v=1}^{n} \frac{\left(\lambda^{k'/\mu'}\right)^{n-v}}{v} \right) = O(1) \left(\frac{1}{n^{k'/\mu'}} \right)$$

which gives us

$$\sum_{n=v}^{\infty} n^{r-1} d_n^{r/k'} |\widehat{t}_{nv}|^{r/\mu} = O(1) \sum_{n=v}^{\infty} \left(\frac{1}{n^{1+r/\mu}} + \frac{\left(\lambda^{r/\mu}\right)^{n-v}}{n} \right) = O(1).$$

Hence $C_\lambda \in B(A_k, A_r)$. □

Lemma 2.7 ([9]). *Suppose that $k > 0$ and $p_n > 0$, $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ as $n \to \infty$. Then there exist two (strictly) positive constants M and N, depending only on k, for which*

$$\frac{M}{P_{v-1}^k} \leq \sum_{n=v}^{\infty} \frac{p_n}{P_n P_{n-1}^k} \leq \frac{N}{P_{v-1}^k}$$

for all $v \geq 1$, where M and N are independent of (p_n).

The p-Cesáro matrix defined in [10] is a triangular matrix T_p with nonzero entries $t_{nv} = 1/(n+1)^p$ for some $p \geq 1$. The case $p = 1$ is reduced to the Cesáro matrix of order one.

Corollary 2.8. *Let T_p be the p-Cesáro matrix and $p > 1$. If $r \geq k \geq 1$, then $T_p \in B(A_k, A_r)$.*

Proof. In Theorem 2.1, take $T = T_p$. Then we have

$$|\widehat{t}_{nv}| = \left| \frac{1}{n^{p-1}} - \frac{1}{(n+1)^{p-1}} - v \left(\frac{1}{n^p} - \frac{1}{(n+1)^p} \right) \right| = O(1) \left(\frac{1}{(n+1)n^{p-1}} + \frac{v}{(n+1)n^p} \right) = O\left(\frac{1}{n^p} \right)$$

for $v \leq n$, and so which gives us

$$\sum_{n=v}^{\infty} n^{r-1} d_n^{r/k'} |\widehat{t}_{nv}|^{r/\mu} = O(1) \sum_{n=v}^{\infty} (\log n)^{r/k'} \frac{1}{n^{(p-1)r+1}} = O(1),$$

for $r \geq k > 1$ by Cauchy Condensation Test (see [11]). Therefore $T_p \in B(A_k, A_r)$.

If T is the matrix of weighted mean $(\overline{N}, p_n)$ (see [1]), then a few calculations reveal that

$$\widehat{t}_{nv} = \frac{p_n P_{v-1}}{P_n P_{n-1}} \quad \text{and} \quad d_n = \left(\frac{p_n}{P_n P_{n-1}} \right)^{k'/\mu'} \sum_{v=1}^{n} \frac{1}{v} P_{v-1}^{k'/\mu'}. \quad \square$$

Corollary 2.9. *Let (p_n) be a positive sequence and let $r \geq k \geq 1$. Then $(\overline{N}, p_n) \in B(A_k, A_r)$ if*

$$np_n = O(P_n) \tag{2.9}$$

and

$$P_n = O(np_n). \tag{2.10}$$

Proof. $d_n = O(1)\left(\frac{p_n}{P_n}\right)^{k'/\mu'}$ for $r \geq k > 1$, by (2.10). So, making use of Lemma 2.7, we get

$$\sum_{n=v}^{\infty} n^{r-1} d_n^{r/k'} |\widehat{t}_{nv}|^{r/\mu} = O(1) P_{v-1}^{r/\mu} \sum_{n=v}^{\infty} \left(\frac{np_n}{P_n}\right)^{r-1} \frac{p_n}{P_n P_{n-1}^{r/\mu}} = O(1)$$

by (2.9). □

Now, using a different technique we give other applications.

Corollary 2.10. *$|\overline{N}, p_n| \Rightarrow |\overline{N}, q_n|_k$, (every series summable $|\overline{N}, p_n|$ is also summable by $|\overline{N}, q_n|_k$), $k \geq 1$, if and only if*

$$\frac{Q_v p_v}{q_v P_v} = O(v^{-1/k'}) \tag{2.11}$$

and

$$\left(\frac{Q_v p_v}{q_v} - P_v\right)\left(\sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{p_n}{P_n P_{n-1}}\right)^k\right)^{1/k} = O(1). \tag{2.12}$$

Proof. In Corollary 2.3, take the matrix $T = t_{nv}$ as follows:

$$t_{nv} = \begin{cases} (p_v/q_v - p_{v+1}/q_{v+1})Q_v/P_n, & \text{if } 0 \leq v \leq n-1 \\ p_n Q_n / P_n q_n, & \text{if } v = n \\ 0, & \text{if } v > n \end{cases}$$

where (p_n) and (q_n) are sequences of positive numbers such that $P_n = p_0 + \cdots + p_n \to \infty$ and $Q_n = q_0 + \cdots + q_n \to \infty$. If (T_n) and (t_n) are sequences of $(\overline{N}, q_n)$ and $(\overline{N}, p_n)$ means of the series $\sum x_v$, then

$$t_n = \sum_{v=0}^{n} t_{nv} T_v.$$

On the other hand, it is easy to see that

$$\widehat{t}_{nv} = \begin{cases} (p_n/P_n P_{n-1})(P_v - Q_v p_v/q_v), & \text{if } 0 \leq v \leq n-1 \\ p_n Q_n / P_n q_n, & \text{if } v = n \\ 0, & \text{if } v > n \end{cases} \tag{2.13}$$

which implies

$$\sum_{n=v}^{\infty} n^{r-1} |\widehat{t}_{nv}|^r = v^{r-1}(p_v Q_v / P_v q_v)^r + |P_v - Q_v p_v / q_v|^r \sum_{n=v+1}^{\infty} n^{r-1}\left(\frac{p_n}{P_n P_{n-1}}\right)^r.$$

Hence the proof is completed by Corollary 2.3. □

This result is the main result of [12].

Corollary 2.11. *Let (p_n) be a positive sequence and $k > 1$. Then $|C, 1|_k \Leftrightarrow |\overline{N}, p_n|_k$ if and only if condition (2.9) and (2.10) is satisfied.*

Proof. Sufficiency. Let $q_n = 1$ in (2.13) and $k = r$ in Theorem 2.1. Then, since $\mu = k = r$ and

$$\widehat{t}_{nv} = \begin{cases} (p_n/P_n P_{n-1})(P_v - (v+1)p_v), & \text{if } 0 \leq v \leq n-1 \\ (v+1)p_v/P_v, & \text{if } v = n \\ 0, & \text{if } v > n \end{cases} \tag{2.14}$$

and so by (2.9) and (2.10) we obtain

$$d_n = \sum_{v=1}^{n} \frac{1}{v} |\widehat{t}_{nv}|^{k'/\mu'} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \frac{1}{v} |P_v - (v+1)p_v| + \frac{(n+1)p_n}{nP_n} = O\left(\frac{1}{n}\right),$$

and so

$$\sum_{n=v}^{\infty} n^{r-1} d_n^{r/k'} |\widehat{t}_{nv}|^{r/\mu} = \frac{(v+1)p_v}{P_v} + |P_v - (v+1)p_v| \sum_{n=v+1}^{\infty} \frac{p_n}{P_n P_{n-1}} = O(1),$$

which gives that $|C,1|_k \Rightarrow |\overline{N},p_n|_k$. Also, if $p_n = 1$ and $r = k$, then we have

$$\widehat{t}_{nv} = \begin{cases} 1/n(n+1)((v+1) - P_v/p_v), & \text{if } 0 \le v \le n-1 \\ P_v/(v+1)p_v, & \text{if } v = n \\ 0, & \text{if } v > n \end{cases}. \tag{2.15}$$

Thus it follows from condition (2.2) and (2.3) of Theorem 2.1 that $|\overline{N},q_n|_k \Rightarrow |C,1|_k$.

Necessity. In Theorem 2.1, take $r = k$. If $|\overline{N},p_n|_k \Rightarrow |C,1|_k$ and $|C,1|_k \Rightarrow |\overline{N},p_n|_k$ then it is seen from (2.14) and (2.15) that

$$\sum_{n=v}^{\infty} n^{k-1}|\widehat{t}_{nv}|^k = v^{k-1}\left(\frac{(v+1)p_v}{P_v}\right)^k |P_v - (v+1)p_v|^k \sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k = O(v^{k-1})$$

and

$$\sum_{n=v}^{\infty} n^{k-1}|\widehat{t}_{nv}|^k = v^{k-1}\left(\frac{P_v}{(v+1)p_v}\right)^k \left|v+1-\frac{P_v}{p_v}\right|^k \sum_{n=v+1}^{\infty} \frac{1}{n(n+1)^k} = O(v^{k-1}),$$

which gives us that (2.11) and (2.12), respectively. □

The sufficiency and necessity of this result are proven in [9,13], respectively.

References

[1] N. Tanović-Miller, On strong summability, Glas. Mat. 34 (1979) 87–97.
[2] G. Das, A Tauberian theorem for absolute summablity, Proc. Cambridge Philos. Soc. 67 (1970) 321–326.
[3] T.M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley, Proc. Lond. Math. Soc. 7 (1957) 113–141.
[4] H. Şevli, E. Savaş, On absolute Cesáro summability, J. Inequal. Appl. (2009) 1–7.
[5] D. Yu, On a result of Flett for Cesáro matrices, Appl. Math. Lett. 22 (2009) 1803–1809.
[6] M.R. Mehdi, Summability factors for generalized absolute summability I, Proc. Lond. Math. Soc. 3 (10) (1960) 180–199.
[7] K. Knopp, G.G. Lorentz, Beitrage zur absoluten limitierung, Arch. Math. 2 (1949) 10–16.
[8] H.C. Rhaly Jr., Discrete generalized Cesáro operators, Proc. Amer. Math. Soc. 86 (1982) 405–409.
[9] M.A. Sarigol, Necessary and sufficient conditions for the equivalence of the summability methods $|\overline{N},p_n|_k$ and $|C,1|_k$, Indian J. Pure Appl. Math. Soc. 22 (6) (1991) 483–489.
[10] H.C. Rhaly Jr., p-Cesáro matrices, Houston J. Math. 15 (1989) 137–146.
[11] J.A. Fridy, Introductory Analysis, Academic Press, California, 2000.
[12] C. Orhan, M.A. Sarigol, On absolute weighted mean summability, Rocky Mountain J. Math. 23 (3) (1993) 1091–1097.
[13] H. Bor, A note on two summability methods, Proc. Amer. Math. Soc. 98 (1986) 81–84.