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Higher order m-point boundary value problems on time scales

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ABSTRACT

In this paper, we investigate the existence of positive solutions for nonlinear even-order m-point boundary value problems on time scales by means of fixed point theorems.

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1. Introduction

The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [1,2]. Motivated by the study of Il'in and Moiseev [1,2], Gupta [3] studied certain three-point boundary value problems for nonlinear ordinary differential equations. Since then, by applying the cone theory techniques, more general nonlinear multi-point boundary value problems have been studied by several authors. We refer the reader to [4–15] and references therein.

The study of dynamic equations on time scales goes back to its founder Hilger [16] and is a rapidly expanding area of research. A result for a dynamic equation contains simultaneously a corresponding result for a differential equation, one for a difference equation, as well as results for other dynamic equations in arbitrary time scales. Some basic definitions and theorems on time scales can be found in the books [17,18]. There are many authors studied the existence of solutions and positive solutions to m-point boundary value problems on time scales. We refer the reader to [19–25]. However, to the best of the author's knowledge, there are no results for positive solutions of higher order m-point boundary value problems on time scales. The aim of this paper is to fill the gap in the relevant literature.

Motivated by Yaslan [26], in this paper, we are concerned with the existence of single and multiple positive solutions to the following nonlinear higher order m-point boundary value problem (BVP) on time scales:

$$\begin{cases} (-1)^n y^{\Delta^{2n}}(t) = f(t, y(t)), \quad t \in [t_1, t_m] \subset \mathbb{T},\ n \in \mathbb{N} \\ y^{\Delta^{2i+1}}(t_1) = 0, \qquad \alpha y^{\Delta^{2i}}(t_m) + \beta y^{\Delta^{2i+1}}(t_m) = \displaystyle\sum_{k=2}^{m-1} y^{\Delta^{2i+1}}(t_k), \end{cases} \tag{1}$$

where $\alpha > 0$ and $\beta > m - 2$ are given constants, $t_1 < t_2 < \cdots < t_{m-1} < t_m$, $m \geq 3$ and $0 \leq i \leq n - 1$. We assume that $f : [t_1, t_m] \times [0, \infty) \to [0, \infty)$ is continuous. Throughout this paper we suppose $\mathbb{T}$ is any time scale and $[t_1, t_m]$ is a subset of $\mathbb{T}$ such that $[t_1, t_m] = \{t \in \mathbb{T} : t_1 \leq t \leq t_m\}$.

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In this paper, we can write $f(t, y(\sigma(t)))$ instead of $f(t, y(t))$ in (1). The presence of the sigma operator in $f(t, y(\sigma(t)))$ does not affect the result.

In this paper, existence results of solutions of BVP (1) are first established as a result of Schauder fixed-point theorem. Second, we establish criteria for the existence of a positive solution of BVP (1) by using Krasnosel'skii fixed-point theorem. Third, we use a result from the theory of fixed point index to show the existence of one or two positive solutions for BVP (1). Fourth, conditions for the existence of at least two positive solutions to BVP (1) are discussed by using Avery–Henderson fixed-point theorem. Finally, we apply the Leggett–Williams fixed-point theorem to prove the existence of at least three positive solutions to BVP (1). The results are even new for the difference equations and differential equations as well as for dynamic equations on general time scales.

2. Preliminaries

We will need the following lemmas, to state the main results of this paper.

Lemma 2.1. *If $\alpha \neq 0$, then Green's function for the boundary value problem*

$$-y^{\Delta^2}(t) = 0, \quad t \in [t_1, t_m],$$
$$y^{\Delta}(t_1) = 0, \qquad \alpha y(t_m) + \beta y^{\Delta}(t_m) = \sum_{k=2}^{m-1} y^{\Delta}(t_k), \quad m \geq 3$$

is given by

$$G(t,s) = \begin{cases} H_1(t,s), & t_1 \leq s \leq \sigma(s) \leq t_2, \\ H_2(t,s), & t_2 \leq s \leq \sigma(s) \leq t_3, \\ \vdots & \\ H_{m-2}(t,s), & t_{m-2} \leq s \leq \sigma(s) \leq t_{m-1}, \\ H_{m-1}(t,s), & t_{m-1} \leq s \leq \sigma(s) \leq t_m, \end{cases} \tag{2}$$

where

$$H_j(t,s) = \begin{cases} t_m + \dfrac{\beta - m + j + 1}{\alpha} - t, & \sigma(s) \leq t, \\ t_m + \dfrac{\beta - m + j + 1}{\alpha} - s, & t \leq s, \end{cases}$$

for all $j = 1, 2, \ldots, m-1$.

Proof. It is easy to see that if $h \in C[t_1, t_m]$, then the following boundary value problem

$$-y^{\Delta^2}(t) = h(t), \quad t \in [t_1, t_m],$$
$$y^{\Delta}(t_1) = 0, \qquad \alpha y(t_m) + \beta y^{\Delta}(t_m) = \sum_{k=2}^{m-1} y^{\Delta}(t_k), \quad m \geq 3$$

has the unique solution

$$\begin{aligned} y(t) &= \int_{t_1}^{t_m} \left(t_m - s + \frac{\beta}{\alpha}\right) h(s)\Delta s - \frac{1}{\alpha} \sum_{k=2}^{m-1} \int_{t_1}^{\sigma(t_k)} h(s)\Delta s + \int_{t_1}^{t} (s-t)h(s)\Delta s \\ &= \int_{t_1}^{t_m} \left(t_m - s + \frac{\beta}{\alpha}\right) h(s)\Delta s - \sum_{j=1}^{m-2} \frac{m-j-1}{\alpha} \int_{t_j}^{t_{j+1}} h(s)\Delta s + \int_{t_1}^{t} (s-t)h(s)\Delta s. \end{aligned}$$

(i) Let $t_j \leq s \leq \sigma(s) \leq t_{j+1}$ for $j = 1, 2, \ldots, m-2$ and $\sigma(s) \leq t$. Then we have

$$G(t,s) = \left(t_m - s + \frac{\beta}{\alpha}\right) - \frac{m-j-1}{\alpha} + (s-t) = t_m + \frac{\beta - m + j + 1}{\alpha} - t.$$

(ii) Let $t_j \leq s \leq \sigma(s) \leq t_{j+1}$ for $j = 1, 2, \ldots, m-2$ and $t \leq s$. Then we obtain

$$G(t,s) = \left(t_m - s + \frac{\beta}{\alpha}\right) - \frac{m-j-1}{\alpha} = t_m + \frac{\beta - m + j + 1}{\alpha} - s.$$

(iii) Assume that $t_{m-1} \leq s \leq \sigma(s) \leq t_m$ and $\sigma(s) \leq t$. Then we get

$$G(t,s) = \left(t_m - s + \frac{\beta}{\alpha}\right) + (s-t) = t_m + \frac{\beta}{\alpha} - t.$$

(iv) Assume that $t_{m-1} \le s \le \sigma(s) \le t_m$ and $t \le s$. Then we have

$$G(t,s) = t_m - s + \frac{\beta}{\alpha}.$$

Hence, we obtain (2). □

Lemma 2.2. *If $\alpha > 0$ and $\beta > m-2$, then Green's function $G(t,s)$ in (2) satisfies the following inequality*

$$G(t,s) \ge \frac{t-t_1}{t_m-t_1} G(t_m,s)$$

for $(t,s) \in [t_1,t_m] \times [t_1,t_m]$.

Proof. (i) Let $s \in [t_1,t_m]$ and $\sigma(s) \le t$. Then we have

$$\frac{G(t,s)}{G(t_m,s)} = \frac{t_m + \frac{\beta-m+j+1}{\alpha} - t}{\frac{\beta-m+j+1}{\alpha}} = 1 + \frac{t_m - t}{\frac{\beta-m+j+1}{\alpha}} > \frac{t-t_1}{t_m-t_1}.$$

(ii) For $s \in [t_1,t_m]$ and $t \le s$, we obtain

$$\frac{G(t,s)}{G(t_m,s)} = 1 \ge \frac{t-t_1}{t_m-t_1}. \quad \square$$

Lemma 2.3. *Let $\alpha > 0$ and $\beta > m-2$. Then Green's function $G(t,s)$ in (2) satisfies*

$$0 < G(t,s) \le G(s,s)$$

for $(t,s) \in [t_1,t_m] \times [t_1,t_m]$.

Proof. Since $\alpha > 0$ and $\beta > m-2$, $H_j(t,s) > 0$ for all $j = 1,2,\ldots,m-1$. Then $G(t,s) > 0$ from (2).
Now, we will show that $G(t,s) \le G(s,s)$.

(i) Let $s \in [t_1,t_m]$ and $\sigma(s) \le t$. Since $G(t,s)$ is decreasing in t, $G(t,s) \le G(s,s)$.
(ii) For $s \in [t_1,t_m]$ and $t \le s$, it is obvious that $G(t,s) = G(s,s)$. □

Lemma 2.4. *Assume that $\alpha > 0$, $\beta > m-2$ and $s \in [t_1,t_m]$. Then Green's function $G(t,s)$ in (2) satisfies*

$$\min_{t\in[t_{m-1},t_m]} G(t,s) \ge K\|G(.,s)\|,$$

where

$$K = \frac{\beta-m+2}{\alpha(t_m-t_1)+\beta-m+2} \tag{3}$$

and $\|.\|$ is defined by $\|x\| = \max_{t\in[t_1,t_m]} |x(t)|$.

Proof. Since Green's function $G(t,s)$ in (2) is nonincreasing in t, we get $\min_{t\in[t_{m-1},t_m]} G(t,s) = G(t_m,s)$. Moreover, from Lemma 2.3 we obtain $\|G(.,s)\| = G(s,s)$ for $s \in [t_1,t_m]$. Then we have

$$G(t_m,s) \ge KG(s,s)$$

from the branches of Green's function $G(t,s)$. □

If we let $G_1(t,s) := G(t,s)$ for G as in (2), then we can recursively define

$$G_j(t,s) = \int_{t_1}^{t_m} G_{j-1}(t,r)G(r,s)\Delta r$$

for $2 \le j \le n$ and $G_n(t,s)$ is Green's function for the homogeneous problem

$$(-1)^n y^{\Delta^{2n}}(t) = 0, \quad t \in [t_1,t_m],$$

$$y^{\Delta^{2i+1}}(t_1) = 0, \qquad \alpha y^{\Delta^{2i}}(t_m) + \beta y^{\Delta^{2i+1}}(t_m) = \sum_{k=2}^{m-1} y^{\Delta^{2i+1}}(t_k),$$

where $m \ge 3$ and $0 \le i \le n-1$.

Lemma 2.5. *Let $\alpha > 0$, $\beta > m-2$. Green's function $G_n(t,s)$ satisfies the following inequalities*

$$0 \le G_n(t,s) \le L^{n-1}\|G(.,s)\|, \quad (t,s) \in [t_1,t_m] \times [t_1,t_m]$$

and

$$G_n(t,s) \geq K^n M^{n-1} \|G(.,s)\|, \quad (t,s) \in [t_{m-1}, t_m] \times [t_1, t_m]$$

where K is given in (3),

$$L = \int_{t_1}^{t_m} \|G(.,s)\| \Delta s > 0 \tag{4}$$

and

$$M = \int_{t_{m-1}}^{t_m} \|G(.,s)\| \Delta s > 0. \tag{5}$$

Proof. Use induction on n and Lemma 2.4. $\square$

Lemma 2.5 has a very important role in the paper and therefore we will assume that $\alpha > 0$ and $\beta > m-2$ throughout the paper.

(1) is equivalent to the nonlinear integral equation

$$y(t) = \int_{t_1}^{t_m} G_n(t,s) f(s, y(s)) \Delta s. \tag{6}$$

Let $\mathcal{B}$ denote the Banach space $C[t_1, t_m]$ with the norm $\|y\| = \max_{t \in [t_1, t_m]} |y(t)|$. Define the cone $P \subset \mathcal{B}$ by

$$P = \left\{ y \in \mathcal{B} : y(t) \geq 0, \min_{t \in [t_{m-1}, t_m]} y(t) \geq \frac{K^n M^{n-1}}{L^{n-1}} \|y\| \right\} \tag{7}$$

where K, L, M are given in (3)–(5), respectively. We can define the operator $A : P \to \mathcal{B}$ by

$$Ay(t) = \int_{t_1}^{t_m} G_n(t,s) f(s, y(s)) \Delta s, \tag{8}$$

where $y \in P$. Therefore solving (6) in P is equivalent to finding fixed points of the operator A.

If $y \in P$, then $Ay(t) \geq 0$ on $[t_1, t_m]$ and by Lemma 2.5 we get

$$\begin{aligned} \min_{t \in [t_{m-1}, t_m]} Ay(t) &= \int_{t_1}^{t_m} \min_{t \in [t_{m-1}, t_m]} G_n(t,s) f(s, y(s)) \Delta s \\ &\geq \frac{K^n M^{n-1}}{L^{n-1}} \int_{t_1}^{t_m} \max_{t \in [t_1, t_m]} |G_n(t,s)| f(s, y(s)) \Delta s \\ &= \frac{K^n M^{n-1}}{L^{n-1}} \|Ay\|. \end{aligned}$$

Thus $Ay \in P$ and therefore $AP \subset P$.

Theorem 2.6 (*Arzela–Ascoli Theorem*)**.** *A set $X \subset C[a,b]$ is relatively compact if and only if the following two conditions are satisfied.*

(a) *The set X is bounded in $C[a,b]$, that is $\|y\| \leq c$ for all $y \in X$.*
(b) *For any given $\varepsilon > 0$, there exists $\delta > 0$ depending only on ε such that for any $y \in X$ and $t_1, t_2 \in [a,b]$ with $|t_1 - t_2| < \delta$, $|y(t_1) - y(t_2)| < \varepsilon$.*

It can be shown that $A : P \to P$ is a completely continuous operator by a standard application of the Arzela–Ascoli theorem.

In order to follow the main results of this paper easily, now we state the fixed point theorems which we applied to prove main theorems.

Theorem 2.7 (*Schauder Fixed Point Theorem*)**.** *Let $\mathcal{B}$ be a Banach space and $\mathcal{S}$ a nonempty bounded, convex, and closed subset of $\mathcal{B}$. Assume that $A : \mathcal{B} \to \mathcal{B}$ is a completely continuous operator. If the operator A leaves the set $\mathcal{S}$ invariant, i.e. if $A(\mathcal{S}) \subset \mathcal{S}$, then A has at least one fixed point in $\mathcal{S}$.*

Theorem 2.8 (*[27] Krasnosel'skii Fixed Point Theorem*)**.** *Let E be a Banach space, and let $K \subset E$ be a cone. Assume that Ω_1 and Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) *$\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$, $\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$;*
or
(ii) *$\|Au\| \geq \|u\|$ for $u \in K \cap \partial\Omega_1$, $\|Au\| \leq \|u\|$ for $u \in K \cap \partial\Omega_2$ hold. Then A has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.*

Definition 2.9. Remember that a subset $K \neq \emptyset$ of X is called a retract of X if there is a continuous map $R : X \to K$, a retraction, such that $Rx = x$ on K. Let X be a Banach space, $K \subset X$ retract, $\Omega \subset K$ open and $f : \overline{\Omega} \to K$ compact and such that $Fix(f) \cap \partial\Omega = \emptyset$. Then we can define an integer $i_K(f, \Omega)$ which has the following properties.

(a) $i_X(f, \Omega) = 1$ for $f(\overline{\Omega}) \in \Omega$.
(b) Let $f : \Omega \to K$ be a continuous function and assume that $Fix(f)$ is a compact subset of Ω. Let Ω_1 and Ω_2 be disjoint open subsets of Ω such that $Fix(f) \subset \Omega_1 \cup \Omega_2$. Then we obtain $i_K(f, \Omega) = i_K(f, \Omega_1) + i_K(f, \Omega_2)$.
(c) Let G be an open subset of $K \times [0, 1]$ and $F : G \to K$ be a continuous map. Assume that $Fix(F)$ is a compact subset of G. If $G_t = \{x : (x, t) \in G\}$ and $F_t = F(., t)$, then we have $i_K(F_0, G_0) = i_K(F_1, G_1)$.
(d) If $K_0 \subset K$ is a retract of K and $F(\overline{\Omega}) \subset K_0$, then $i_K(F, \Omega) = i_{K_0}(F, \Omega \cap K_0)$.

We will apply the following well-known result of the fixed point theorems to prove the existence of one or two positive solutions to (1).

Lemma 2.10 ([27,28])**.** *Let P be a cone in a Banach space $\mathcal{B}$, and let D be an open, bounded subset of $\mathcal{B}$ with $D_P := D \cap P \neq \emptyset$ and $\overline{D}_P \neq P$. Assume that $A : \overline{D}_P \to P$ is a compact map such that $y \neq Ay$ for $y \in \partial D_P$. The following results hold.*

(i) *If $\|Ay\| \le \|y\|$ for $y \in \partial D_P$, then $i_P(A, D_P) = 1$.*
(ii) *If there exists a $b \in P \setminus \{0\}$ such that $y \neq Ay + \lambda b$ for all $y \in \partial D_P$ and all $\lambda > 0$, then $i_P(A, D_P) = 0$.*
(iii) *Let U be open in P such that $\overline{U}_P \subset D_P$. If $i_P(A, D_P) = 1$ and $i_P(A, U_P) = 0$, then A has a fixed point in $D_P \setminus \overline{U}_P$. The same result holds if $i_P(A, D_P) = 0$ and $i_P(A, U_P) = 1$.*

Theorem 2.11 ([29] *Avery–Henderson Fixed Point Theorem*)**.** *Let P be a cone in a real Banach space E. Set*

$$P(\phi, r) = \{u \in P : \phi(u) < r\}.$$

Assume that there exist positive numbers r and M, nonnegative increasing continuous functionals η, ϕ on P, and a nonnegative continuous functional θ on P with $\theta(0) = 0$ such that

$$\phi(u) \le \theta(u) \le \eta(u) \quad \text{and} \quad \|u\| \le M\phi(u)$$

for all $u \in \overline{P(\phi, r)}$. Suppose that there exist positive numbers $p < q < r$ such that

$$\theta(\lambda u) \le \lambda\theta(u), \quad \text{for all } 0 \le \lambda \le 1 \text{ and } u \in \partial P(\theta, q).$$

If $A : \overline{P(\phi, r)} \to P$ is a completely continuous operator satisfying

(i) *$\phi(Au) > r$ for all $u \in \partial P(\phi, r)$,*
(ii) *$\theta(Au) < q$ for all $u \in \partial P(\theta, q)$,*
(iii) *$P(\eta, p) \neq \emptyset$ and $\eta(Au) > p$ for all $u \in \partial P(\eta, p)$,*

then A has at least two fixed points u_1 and u_2 such that

$$p < \eta(u_1) \quad \text{with } \theta(u_1) < q \text{ and } q < \theta(u_2) \quad \text{with } \phi(u_2) < r.$$

Theorem 2.12 ([30] *Leggett–Williams Fixed Point Theorem*)**.** *Let P be a cone in a real Banach space E. Set*

$$P_r := \{x \in P : \|x\| < r\}$$
$$P(\psi, a, b) := \{x \in P : a \le \psi(x),\ \|x\| \le b\}.$$

Suppose $A : \overline{P_r} \to \overline{P_r}$ be a completely continuous operator and ψ be a nonnegative continuous concave functional on P with $\psi(u) \le \|u\|$ for all $u \in \overline{P_r}$. If there exist $0 < p < q < l \le r$ such that the following conditions hold:

(i) *$\{u \in P(\psi, q, l) : \psi(u) > q\} \neq \emptyset$ and $\psi(Au) > q$ for all $u \in P(\psi, q, l)$;*
(ii) *$\|Au\| < p$ for $\|u\| \le p$;*
(iii) *$\psi(Au) > q$ for $u \in P(\psi, q, r)$ with $\|Au\| > l$,*

then A has at least three fixed points u_1, u_2 and u_3 in $\overline{P_r}$ satisfying

$$\|u_1\|\langle p,\ \psi(u_2)\rangle q, \quad p < \|u_3\| \text{ with } \psi(u_3) < q.$$

3. Main results

Theorem 3.1. *Assume $\alpha > 0$, $\beta > m - 2$. Let there exist a number $R > 0$ such that $NL^n \le R$, where $N \ge \max_{\|y\| \le R} |f(t, y(t))|$, for $t \in [t_1, t_m]$ and L is as in (4). Then BVP (1) has at least one solution $y(t)$.*

Proof. Using the Schauder fixed point theorem, the proof is very similar to the proof of Theorem 1 in [26] and is omitted. □

Theorem 3.2. *Assume $\alpha > 0$, $\beta > m-2$. In addition, let there exist numbers $0 < r < R < \infty$ such that*

$$f(t,y) < \frac{1}{L^n}y, \quad \text{if } 0 \le y \le r$$

and

$$f(t,y) > \frac{L^{n-1}}{K^{2n}M^{2n-1}}y, \quad \text{if } R \le y < \infty$$

for $t \in [t_1, t_m]$, where K, L, M are as in (3)–(5), respectively. Then BVP (1) has at least one positive solution.

Proof. Let us now set

$$\Omega_1 := \{y \in P : \|y\| < r\}.$$

If $y \in P \cap \partial\Omega_1$, then from Lemma 2.5 we obtain

$$\begin{aligned} Ay(t) &= \int_{t_1}^{t_m} G_n(t,s)f(s,y(s))\Delta s \\ &< \frac{1}{L^n}\int_{t_1}^{t_m} G_n(t,s)y(s)\Delta s \\ &\le \frac{1}{L}\|y\|\int_{t_1}^{t_m} \|G(.,s)\|\Delta s \\ &= \|y\| \end{aligned}$$

for $t \in [t_1, t_m]$. Thus, we get $\|Ay\| \le \|y\|$ for $y \in P \cap \partial\Omega_1$.

If we let

$$\Omega_2 := \left\{y \in P : \|y\| < \frac{L^{n-1}}{K^nM^{n-1}}R\right\},$$

then for $y \in P$ with $\|y\| = \frac{L^{n-1}}{K^nM^{n-1}}R$, we have

$$y(t) \ge \frac{K^nM^{n-1}}{L^{n-1}}\|y\| = R$$

for $t \in [t_1, t_m]$. Therefore from Lemma 2.5, we have

$$\begin{aligned} Ay(t) &= \int_{t_1}^{t_m} G_n(t,s)f(s,y(s))\Delta s \\ &> \frac{L^{n-1}}{K^{2n}M^{2n-1}}\int_{t_{m-1}}^{t_m} G_n(t,s)y(s)\Delta s \\ &\ge \frac{1}{K^nM^n}\|y\|\int_{t_{m-1}}^{t_m} G_n(t,s)\Delta s \\ &\ge \|y\|. \end{aligned}$$

Hence, $\|Ay\| \ge \|y\|$ for $y \in P \cap \partial\Omega_2$. Thus, by (i) of Theorem 2.8, A has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$, such that $r \le \|y\| \le \frac{L^{n-1}}{k^nM^{n-1}}R$. Therefore, BVP (1) has at least one positive solution. □

Now we will investigate the existence of one or two positive solutions for BVP (1) by using Lemma 2.10.

For the cone P given in (7) and any positive real number r, define the convex set

$$P_r := \{y \in P : \|y\| < r\}$$

and the set

$$\Omega_r := \{y \in P : \min_{t\in[t_{m-1},t_m]} y(t) < er\}$$

where

$$e := \frac{K^nM^{n-1}}{L^{n-1}} \in (0,1) \tag{9}$$

and K, L, and M are defined in (3)–(5), respectively. The following results are proved in [28].

Lemma 3.3. *The set Ω_r has the following properties.*

(i) *Ω_r is open relative to P.*
(ii) *$P_{er} \subset \Omega_r \subset P_r$*
(iii) *$y \in \partial\Omega_r$ if and only if $\min_{t\in[t_{m-1},t_m]} y(t) = er$.*
(iv) *If $y \in \partial\Omega_r$, then $er \le y(t) \le r$ for $t \in [t_{m-1}, t_m]$.*

For convenience, we introduce the following notations. Let

$$f_{er}^r := \min\left\{\min_{t\in[t_{m-1},t_m]} \frac{f(t,y)}{r} : y \in [er, r]\right\}$$

$$f_0^r := \max\left\{\max_{t\in[t_1,t_m]} \frac{f(t,y)}{r} : y \in [0, r]\right\}$$

$$f^a := \limsup_{y\to a} \max_{t\in[t_1,t_m]} \frac{f(t,y)}{y}$$

$$f_a := \liminf_{y\to a} \min_{t\in[t_{m-1},t_m]} \frac{f(t,y)}{y} \quad (a := 0^+, \infty).$$

In the next two lemmas, we give conditions on f guaranteeing that $i_P(A, P_r) = 1$ or $i_P(A, \Omega_r) = 0$.

Lemma 3.4. *Let $\alpha > 0$ and $\beta > m - 2$. For L in (4), if the conditions*

$$f_0^r \le \frac{1}{L^n} \quad \text{and} \quad y \ne Ay \quad \text{for } y \in \partial P_r,$$

hold, then $i_P(A, P_r) = 1$.

Proof. If $y \in \partial P_r$, then using Lemma 2.5, we have

$$\begin{aligned} Ay(t) &= \int_{t_1}^{t_m} G_n(t,s) f(s, y(s))\Delta s \\ &\le \|f(.,y)\| L^{n-1} \int_{t_1}^{t_m} \|G(.,s)\|\Delta s \\ &\le \frac{r}{L^n} L^n = r = \|y\|. \end{aligned}$$

It follows that $\|Ay\| \le \|y\|$ for $y \in \partial P_r$. By Lemma 2.10(i), we get $i_P(A, P_r) = 1$. □

Lemma 3.5. *Let $\alpha > 0$, $\beta > m - 2$ and*

$$N := \left(\int_{t_{m-1}}^{t_m} \min_{t\in[t_{m-1},t_m]} G_n(t,s)\Delta s\right)^{-1}. \tag{10}$$

If the conditions

$$f_{er}^r \ge Ne \quad \text{and} \quad y \ne Ay \text{ for } y \in \partial\Omega_r$$

hold, then $i_P(A, \Omega_r) = 0$.

Proof. Let $b(t) \equiv 1$ for $t \in [t_1, t_m]$, then $b \in \partial P_1$. Assume that there exist $y_0 \in \partial\Omega_r$ and $\lambda_0 > 0$ such that $y_0 = Ay_0 + \lambda_0 b$. Then for $t \in [t_{m-1}, t_m]$ we have

$$\begin{aligned} y_0(t) &= Ay_0(t) + \lambda_0 b(t) \ge \int_{t_{m-1}}^{t_m} G_n(t,s) f(s, y_0(s))\Delta s + \lambda_0 \\ &\ge Ner \int_{t_{m-1}}^{t_m} \min_{t\in[t_{m-1},t_m]} G_n(t,s)\Delta s + \lambda_0 \\ &= er + \lambda_0. \end{aligned}$$

But this implies that $er \ge er + \lambda_0$, a contradiction. Hence, $y_0 \ne Ay_0 + \lambda_0 b$ for $y_0 \in \partial\Omega_r$ and $\lambda_0 > 0$, so by Lemma 2.10(ii), we get $i_P(A, \Omega_r) = 0$. □

Theorem 3.6. *Assume that $\alpha > 0$ and $\beta > m - 2$. Let L, e, and N be as in (4), (9), (10), respectively. Suppose that one of the following conditions holds.*

(C1) *There exist constants* $c_1, c_2, c_3 \in \mathbb{R}$ *with* $0 < c_1 < c_2 < ec_3$ *such that*

$$f_{ec_1}^{c_1}, f_{ec_3}^{c_3} \geq Ne, f_0^{c_2} \leq \frac{1}{L^n}, \quad \text{and} \quad y \neq Ay \text{ for } y \in \partial P_{c_2}.$$

(C2) *There exist constants* $c_1, c_2, c_3 \in \mathbb{R}$ *with* $0 < c_1 < ec_2$ *and* $c_2 < c_3$ *such that*

$$f_0^{c_1}, f_0^{c_3} \leq \frac{1}{L^n}, f_{ec_2}^{c_2} \geq Ne, \quad \text{and} \quad y \neq Ay \text{ for } y \in \partial \Omega_{c_2}.$$

Then (1) *has two positive solutions. Additionally, if in* (C2) *the condition* $f_0^{c_1} \leq \frac{1}{L^n}$ *is replaced by* $f_0^{c_1} < \frac{1}{L^n}$, *then* (1) *has a third positive solution in* P_{c_1}.

Proof. Assume that (C1) holds. We show that either A has a fixed point in $\partial\Omega_{c_1}$ or in $P_{c_2} \setminus \overline{\Omega}_{c_1}$. If $y \neq Ay$ for $y \in \partial\Omega_{c_1}$, then by Lemma 3.5, we have $i_P(A, \Omega_{c_1}) = 0$. Since $f_0^{c_2} \leq \frac{1}{L^n}$ and $y \neq Ay$ for $y \in \partial P_{c_2}$, from Lemma 3.4 we get $i_P(A, P_{c_2}) = 1$. By Lemma 3.3(ii) and $c_1 < c_2$, we have $\overline{\Omega}_{c_1} \subset \overline{P}_{c_1} \subset P_{c_2}$. From Lemma 2.10(iii), A has a fixed point in $P_{c_2} \setminus \overline{\Omega}_{c_1}$. If $y \neq Ay$ for $y \in \partial\Omega_{c_3}$, then $i_P(A, \Omega_{c_3}) = 0$ from Lemma 3.5. By Lemma 3.3(ii) and $c_2 < ec_3$, we get $\overline{P}_{c_2} \subset P_{ec_3} \subset \Omega_{c_3}$. From Lemma 2.10(iii), A has a fixed point in $\Omega_{c_3} \setminus \overline{P}_{c_2}$. The proof is similar when (C2) holds and we omit it here. □

Corollary 3.7. *Assume that* $\alpha > 0$ *and* $\beta > m - 2$. *Let there exist a constant* $c > 0$ *such that one of the following conditions holds.*

(H1) $N < f_0, f_\infty \leq \infty, f_0^c \leq \frac{1}{L^n}$, *and* $y \neq Ay$ *for* $y \in \partial P_c$.
(H2) $0 \leq f^0, f^\infty < \frac{1}{L^n}, f_{ec}^c \geq Ne$, *and* $y \neq Ay$ *for* $y \in \partial\Omega_c$.

Then (1) *has two positive solutions.*

Proof. Since (H1) implies (C1) and (H2) implies (C2), the result follows. □

As a special case of Theorem 3.6 and Corollary 3.7, we have the following two results.

Theorem 3.8. *Let* $\alpha > 0$ *and* $\beta > m - 2$. *Assume that one of the following conditions holds.*

(C3) *There exist constants* $c_1, c_2 \in \mathbb{R}$ *with* $0 < c_1 < c_2$ *such that*

$$f_{ec_1}^{c_1} \geq Ne \quad \text{and} \quad f_0^{c_2} \leq \frac{1}{L^n}.$$

(C4) *There exist constants* $c_1, c_2 \in \mathbb{R}$ *with* $0 < c_1 < ec_2$ *such that*

$$f_0^{c_1} \leq \frac{1}{L^n} \quad \text{and} \quad f_{ec_2}^{c_2} \geq Ne.$$

Then (1) *has a positive solution.*

Corollary 3.9. *Let* $\alpha > 0$ *and* $\beta > m - 2$. *Assume that one of the following conditions holds.*

(H3) $0 \leq f^\infty < \frac{1}{L^n}$ *and* $N < f_0 \leq \infty$.
(H4) $0 \leq f^0 < \frac{1}{L^n}$ *and* $N < f_\infty \leq \infty$.

Then (1) *has a positive solution.*

Now we will give the sufficient conditions to have at least two positive solutions for BVP (1). The Avery–Henderson fixed point theorem will be used to prove the result.

Theorem 3.10. *Assume* $\alpha > 0$, $\beta > m - 2$. *Suppose there exist numbers* $0 < p < q < r$ *such that the function* f *satisfies the following conditions:*

(i) $f(t, y) > \frac{r}{K^n M^n}$ *for* $t \in [t_{m-1}, t_m]$ *and* $y \in [r, \frac{r}{e}]$;
(ii) $f(t, y) < \frac{q}{L^n}$ *for* $t \in [t_1, t_m]$ *and* $y \in [0, \frac{q}{e}]$;
(iii) $f(t, y) > \frac{p}{K^n M^n}$ *for* $t \in [t_{m-1}, t_m]$ *and* $y \in [ep, p]$,

where K, L, M, *and* e *are defined in* (3)–(5) *and* (9), *respectively. Then BVP* (1) *has at least two positive solutions* y_1 *and* y_2 *such that*

$$p < \max_{t \in [t_1, t_m]} y_1(t) \quad \text{with} \quad \max_{t \in [t_{m-1}, t_m]} y_1(t) < q$$
$$q < \max_{t \in [t_{m-1}, t_m]} y_2(t) \quad \text{with} \quad \min_{t \in [t_{m-1}, t_m]} y_2(t) < r.$$

Proof. Define the cone P as in (7). From Lemma 2.5, $AP \subset P$ and A is completely continuous. Let the nonnegative increasing continuous functionals ϕ, θ and η be defined on the cone P by

$$\phi(y) := \min_{t\in[t_{m-1},t_m]} y(t), \qquad \theta(y) := \max_{t\in[t_{m-1},t_m]} y(t), \qquad \eta(y) := \max_{t\in[t_1,t_m]} y(t).$$

For each $y \in P$, we have

$$\phi(y) \le \theta(y) \le \eta(y)$$

and from (7)

$$\|y\| \le \frac{1}{e}\phi(y).$$

Moreover, $\theta(0) = 0$ and for all $y \in P$, $\lambda \in [0, 1]$ we get $\theta(\lambda y) = \lambda\theta(y)$.

We now verify that the remaining conditions of Theorem 2.11 hold. □

Claim 1. *If $y \in \partial P(\phi, r)$, then $\phi(Ay) > r$: Since $y \in \partial P(\phi, r)$, we have $r = \min_{t\in[t_{m-1},t_m]} y(t) \le \|y\| \le \frac{r}{e}$ for $t \in [t_{m-1}, t_m]$. Then, using hypothesis* (i) *and* Lemma 2.5 *we obtain*

$$\begin{aligned}\phi(Ay) &= \int_{t_1}^{t_m} \min_{t\in[t_{m-1},t_m]} G_n(t,s)f(s,y(s))\Delta s\\ &\ge K^n M^{n-1}\int_{t_{m-1}}^{t_m} \|G(.,s)\| f(s,y(s))\Delta s\\ &> r.\end{aligned}$$

Claim 2. *If $y \in \partial P(\theta, q)$, then $\theta(Ay) < q$: since $y \in \partial P(\theta, q)$, $0 \le y(t) \le \|y\| \le \frac{q}{e}$ for $t \in [t_1, t_m]$. Thus, by hypothesis* (ii) *and* Lemma 2.5 *we have*

$$\begin{aligned}\theta(Ay) &= \int_{t_1}^{t_m} \max_{t\in[t_{m-1},t_m]} G_n(t,s)f(s,y(s))\Delta s\\ &\le L^{n-1}\int_{t_1}^{t_m} \|G(.,s)\| f(s,y(s))\Delta s\\ &< q.\end{aligned}$$

Claim 3. *$P(\eta, p) \ne \emptyset$ and $\eta(Ay) > p$ for all $y \in \partial P(\eta, p)$: since $\frac{p}{2} \in P$ and $p > 0$, $P(\eta, p) \ne \emptyset$. If $y \in \partial P(\eta, p)$, we get $ep \le y(t) \le \|y\| = p$ for $t \in [t_{m-1}, t_m]$. Hence, using hypothesis* (iii) *and* Lemma 2.5 *we obtain*

$$\begin{aligned}\eta(Ay) &\ge \int_{t_1}^{t_m} G_n(t,s)f(s,y(s))\Delta s\\ &\ge K^n M^{n-1}\int_{t_{m-1}}^{t_m} \|G(.,s)\| f(s,y(s))\Delta s\\ &> p.\end{aligned}$$

Since the conditions of Theorem 2.11 *are satisfied, BVP* (1) *has at least two positive solutions y_1 and y_2 such that*

$$\begin{aligned}&p < \max_{t\in[t_1,t_m]} y_1(t) \quad \text{with} \quad \max_{t\in[t_{m-1},t_m]} y_1(t) < q\\ &q < \max_{t\in[t_{m-1},t_m]} y_2(t) \quad \text{with} \quad \min_{t\in[t_{m-1},t_m]} y_2(t) < r. \quad \square\end{aligned}$$

Now, we will apply the Leggett–Williams fixed point theorem to prove the following theorem.

Theorem 3.11. *Let $\alpha > 0$, $\beta > m - 2$. Suppose that there exist numbers $0 < p < q < \frac{q}{e} \le r$ such that the function f satisfies the following conditions:*

(i) $f(t,y) \le \frac{r}{L^n}$ *for $t \in [t_1, t_m]$ and $y \in [0, r]$,*
(ii) $f(t,y) > \frac{q}{K^n M^n}$ *for $t \in [t_{m-1}, t_m]$ and $y \in [q, \frac{q}{e}]$,*
(iii) $f(t,y) < \frac{p}{L^n}$ *for $t \in [t_1, t_m]$ and $y \in [0, p]$,*

where K, L, M, and e are as defined in (3)–(5) *and* (9)*, respectively. Then* (1) *has at least three positive solutions y_1, y_2 and y_3 satisfying*

$$\max_{t\in[t_1,t_m]} y_1(t) < p, \qquad q < \min_{t\in[t_{m-1},t_m]} y_2(t), \qquad p < \max_{t\in[t_1,t_m]} y_3(t) \quad \text{with} \quad \min_{t\in[t_{m-1},t_m]} y_3(t) < q.$$

Proof. Define the nonnegative continuous concave functional $\psi : P \to [0, \infty)$ to be $\psi(y) := \min_{t\in[t_{m-1},t_m]} y(t)$ and the cone P as in (7). For all $y \in P$, we have $\psi(y) \le \|y\|$. If $y \in \overline{P_r}$, then $0 \le y \le r$ and $f(t, y) \le \frac{r}{L^n}$ from hypothesis (i). Then we get

$$\begin{aligned}\|Ay\| &= \int_{t_1}^{t_m} \max_{t\in[t_1,t_m]} G_n(t, s)f(s, y(s))\Delta s\\ &\le L^{n-1}\int_{t_1}^{t_m} \|G(., s)\| f(s, y(s))\Delta s\\ &\le r\end{aligned}$$

by Lemma 2.5. This proves that $A : \overline{P_r} \to \overline{P_r}$.

Since $K < 1$ and $\frac{M}{L} < 1$, $y(t) \equiv \frac{q}{e} \in P(\psi, q, \frac{q}{e})$ and $\psi(\frac{q}{e}) > q$. Then $\{y \in P(\psi, q, \frac{q}{e}) : \psi(y) > q\} \ne \emptyset$. For all $y \in P(\psi, q, \frac{q}{e})$, we have $q \le \min_{t\in[t_{m-1},t_m]} y(t) \le \|y\| \le \frac{q}{e}$ for $t \in [t_{m-1}, t_m]$. Using hypothesis (ii) and Lemma 2.5, we find

$$\begin{aligned}\psi(Ay) &= \int_{t_1}^{t_m} \min_{t\in[t_{m-1},t_m]} G_n(t, s)f(s, y(s))\Delta s\\ &\ge K^nM^{n-1}\int_{t_{m-1}}^{t_m} \|G(., s)\| f(s, y(s))\Delta s\\ &> q.\end{aligned}$$

Hence, condition (i) of Theorem 2.12 holds.

If $\|y\| \le p$, then $f(t, y) < \frac{p}{L^n}$ for $t \in [t_1, t_m]$ from hypothesis (iii). We obtain

$$\begin{aligned}\|Ay\| &= \int_{t_1}^{t_m} \max_{t\in[t_1,t_m]} G_n(t, s)f(s, y(s))\Delta s\\ &\le L^{n-1}\int_{t_1}^{t_m} \|G(., s)\| f(s, y(s))\Delta s\\ &< p.\end{aligned}$$

Consequently, condition (ii) of Theorem 2.12 is satisfied.

For condition (iii) of Theorem 2.12, we suppose that $y \in P(\psi, q, r)$ with $\|Ay\| > \frac{q}{e}$. Then, from Lemma 2.5 we obtain

$$\psi(Ay) = \min_{t\in[t_{m-1},t_m]} Ay(t) \ge \frac{K^nM^{n-1}}{L^{n-1}}\|Ay\| > q.$$

This completes the proof. □

Using the ideas in the proof of the above problem, we can establish the existence of an arbitrary odd number of positive solutions of (1).

Theorem 3.12. *Let $\alpha > 0$, $\beta > m - 2$. Suppose that there exist numbers*

$$0 < p_1 < q_1 < \frac{q_1}{e} \le p_2 < q_2 < \frac{q_2}{e} \le p_3 < \cdots \le p_n, \quad n \in 2, 3, \ldots$$

such that the function f satisfies the following conditions:

(i) $f(t, y) < \frac{p_i}{L^n}$ *for $t \in [t_1, t_m]$ and $y \in [0, p_i]$,*
(ii) $f(t, y) > \frac{q_i}{K^nM^n}$ *for $t \in [t_{m-1}, t_m]$ and $y \in [q_i, \frac{q_i}{e}]$,*

where K, L, M, and e are as defined in (3)–(5) and (9), respectively. Then m-point BVP (1) has at least $2n - 1$ positive solutions.

Proof. Use induction on n. □

Example 3.13. Let $\mathbb{T} = \{(\frac{1}{3})^n : n \in \mathbb{N}_0\} \cup \{0\}$. Consider the following boundary value problem on $\mathbb{T}$:

$$\begin{cases} y^{\Delta^4}(t) = \dfrac{2(y+5)^2}{y^4+4}, \quad t \in [0, 1] \subset \mathbb{T}\\ y^{\Delta}(0) = 0, \quad y(1) + 3y^{\Delta}(1) = y^{\Delta}\left(\dfrac{1}{27}\right) + y^{\Delta}\left(\dfrac{1}{9}\right),\\ y^{\Delta^3}(0) = 0, \quad y^{\Delta^2}(1) + 3y^{\Delta^3}(1) = y^{\Delta^3}\left(\dfrac{1}{27}\right) + y^{\Delta^3}\left(\dfrac{1}{9}\right),\end{cases}$$

where $t_1 = 0, t_2 = \frac{1}{27}, t_3 = \frac{1}{9}, t_4 = 1 = \alpha, \beta = 3, n = 2, m = 4$ and $f(t, y) = \frac{2(y+5)^2}{y^4+4}$. Green's function $G(t, s)$ is

$$G(t,s) = \begin{cases} H_1(t,s), & 0 \le s \le \sigma(s) \le \dfrac{1}{27}, \\ H_2(t,s), & \dfrac{1}{27} \le s \le \sigma(s) \le \dfrac{1}{9}, \\ H_3(t,s), & \dfrac{1}{9} \le s \le \sigma(s) \le 1, \end{cases}$$

where

$$H_1(t,s) = \begin{cases} 2 - t, & \sigma(s) \le t, \\ 2 - s, & t \le s, \end{cases}$$

$$H_2(t,s) = \begin{cases} 3 - t, & \sigma(s) \le t, \\ 3 - s, & t \le s, \end{cases}$$

and

$$H_3(t,s) = \begin{cases} 4 - t, & \sigma(s) \le t, \\ 4 - s, & t \le s. \end{cases}$$

Then we obtain $K = \frac{1}{2}, L = \frac{173}{36}, M = 4$ and $e = \frac{36}{173}$.

If we take $p = 1, q = 1.5$ and $r = 3$, then $0 < p < q < r$ and conditions (i)–(iii) of Theorem 3.10 are satisfied. Hence, the BVP has at least two positive solutions y_1 and y_2 satisfying

$$p < \max_{t \in [0,1]} y_1(t) \quad \text{with} \quad \max_{t \in [\frac{1}{9},1]} y_1(t) < q$$

$$q < \max_{t \in [\frac{1}{9},1]} y_2(t) \quad \text{with} \quad \min_{t \in [\frac{1}{9},1]} y_2(t) < r.$$

If we take $p = 0.5, q = 1$ and $r = 5.5$, then $0 < p < q < \frac{q}{e} \le r$ and conditions (i)–(iii) of Theorem 3.11 are satisfied. Hence, the BVP has at least three positive solutions y_1, y_2 and y_3 satisfying

$$\max_{t \in [0,1]} y_1(t) < p, \qquad q < \min_{t \in [\frac{1}{9},1]} y_2(t), \qquad p < \max_{t \in [0,1]} y_3(t) \quad \text{with} \quad \min_{t \in [\frac{1}{9},1]} y_3(t) < q.$$

If we take $p_1 = 0.01, q_1 = 0.02, p_2 = 0.098, q_2 = 0.1, p_3 = 0.48, q_3 = 0.5$ and $p_4 = 5.5$ then $0 < p_1 < q_1 < \frac{q_1}{e} \le p_2 < q_2 < \frac{q_2}{e} \le p_3 < q_3 < \frac{q_3}{e} \le p_4$ and conditions (i), (ii) of Theorem 3.12 are satisfied. Thus, the BVP has at least seven positive solutions.

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