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EXISTENCE OF POSITIVE SOLUTIONS FOR MULTI-POINT TIME SCALE BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS

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Abstract. In this paper, we establish the criteria for the existence of at least one and three positive solutions for a nonlinear second order multi-point time scale boundary value problem on infinite interval based on the Leray-Schauder fixed point theorem and the five functional fixed point theorem, respectively.

Keywords. Boundary value problem; Cone; Fixed point theorem; Positive solution; Time scales.

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1. Introduction

Let $\mathbb{T}$ be any time scale. We consider the following nonlinear second order multi-point time scale boundary value problem (BVP) on infinite interval:

$$\begin{cases} \frac{1}{p(t)}(p(t)u^{\Delta}(t))^{\Delta} + f(t,u(t),u^{\Delta}(t)) = 0, \ t \in [t_1,\infty)_{\mathbb{T}}, \\ \alpha u(t_1) - \beta p(t_1)u^{\Delta}(t_1) = \sum\limits_{j=1}^{m-2} a_j p(\xi_j)u^{\Delta}(\xi_j), \\ \delta \lim_{t\to\infty} p(t)u^{\Delta}(t) = \sum\limits_{j=1}^{m-2} b_j p(\xi_j)u^{\Delta}(\xi_j), \end{cases} \tag{1.1}$$

where $m \geq 3$, $\alpha, \beta, \delta > 0$, $\delta > \sum\limits_{j=1}^{m-2} b_j$, $0 \leq t_1 < \xi_1 < \ldots < \xi_{m-2} < \infty$ and $a_j, b_j \geq 0$ $(1 \leq j \leq m-2)$ are given constants. We assume that $p \in C([t_1,\infty)_{\mathbb{T}},(0,\infty))$ and $\int\limits_{t_1}^{\infty} \frac{\Delta\tau}{p(\tau)} < \infty$.

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Throughout this paper, we always suppose that $\mathbb{T}$ is any time scale (nonempty closed subset of $\mathbb{R}$). The study of dynamic equations on time scales goes back to its founder Hilger [8] and is a rapidly expanding area of research. An important class of dynamic equations is boundary value problems, due to their striking applications to almost all area of sciences, engineering and technology. By researching boundary value problems on time scales the results unify the theory of differential and difference equations (and removes obscurity from both areas) and provide accurate information of phenomena that manifest themselves partly in continuous time and partly in discrete time. Some basic definitions and theorems on time scales can be found in the books [3, 4], which are excellent references for calculus of time scales.

The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [9, 10]. Much of theory of time scale dynamic equations on finite intervals have been presented in [5, 7, 13, 14, 15, 16, 17, 20, 21, 18, 19, 22, 23] and references therein. However, there is significantly less literature available on the basic theory of time scale dynamic equations on infinite intervals. Due to the fact that an infinite interval is noncompact, the discussion about boundary value problem on the infinite intervals is more complicated. The study of time scale boundary value problems on infinite intervals was initiated by Agarwal, Bohner and O'Regan [1]. Since then, there are a few authors studied the existence of positive solutions for time scale boundary value problems on infinite intervals. We refer the reader to [6, 11, 12, 25, 26] and references therein.

In this paper, first, we provide some preliminary lemmas which are key tools for our main results. Second, we obtain the existence of at least one positive solution for the BVP (1.1) by using the Leray-Schauder fixed point theorem. Finally, we use the five functional fixed-point theorem to show that the existence of at least three positive solutions to the BVP (1.1).

To the best of our knowledge, the existence results for positive solutions of the BVP (1.1) have not been studied previously. The results are even new for the difference equations and differential equations as well as for dynamic equations on general time scales.

Throughout the paper, we assume that the following condition is satisfied:

(H1) $f \in C([t_1,\infty)_{\mathbb{T}} \times [0,\infty) \times [0,\infty), [0,\infty))$, $h \in C([0,\infty) \times [0,\infty), [0,\infty))$ and for all $t \in [t_1,\infty)_{\mathbb{T}}$, $f(t,(1+t)x,y) \leq \vartheta(t)h((1+t)x,y)$, where $\vartheta : [t_1,\infty)_{\mathbb{T}} \to (0,\infty)$ is continuous and $\int_{t_1}^{\infty} \vartheta(s)p(s)\Delta s < \infty$.

2. Preliminaries

To state the main results of this paper, we need several lemmas. Let us define $\theta(t)$ and $\varphi(t)$ the solutions of the corresponding homogeneous equation

$$\frac{1}{p(t)}(p(t)u^{\Delta}(t))^{\Delta}=0 \tag{2.1}$$

under the initial conditions

$$\begin{aligned}&\theta(t_1)=\beta,\, p(t_1)\theta^{\Delta}(t_1)=\alpha,\\ &\varphi(\infty)=\delta,\, p(\infty)\varphi^{\Delta}(\infty)=0.\end{aligned} \tag{2.2}$$

Using initial conditions (2.2), we can deduce the following equations:

$$\theta(t)=\beta+\alpha\int_{t_1}^{t}\frac{\Delta\tau}{p(\tau)}, \tag{2.3}$$

$$\varphi(t)=\delta$$

from equation (2.1) for $\theta(t)$ and $\varphi(t)$. Let us define $D:=\alpha\delta$. The Green's function for the homogeneous problem corresponding to the BVP (1.1) is given by

$$G(t,s)=\frac{1}{D}\begin{cases}\theta(t)\varphi(s),\, t_1\le t\le s<\infty\\ \theta(s)\varphi(t),\, t_1\le s\le t<\infty\end{cases}=\begin{cases}\frac{\beta}{\alpha}+\int_{t_1}^{t}\frac{\Delta\tau}{p(\tau)},\, t_1\le t\le s<\infty,\\ \frac{\beta}{\alpha}+\int_{t_1}^{s}\frac{\Delta\tau}{p(\tau)},\, t_1\le s\le t<\infty.\end{cases} \tag{2.4}$$

Lemma 2.1. *If $D>0$, then solution of the BVP (1.1) is*

$$u(t)=\int_{t_1}^{\infty}G(t,s)p(s)f(s,u(s),u^{\Delta}(s))\Delta s+\frac{C}{D}\theta(t)+\frac{B}{D}\varphi(t), \tag{2.5}$$

where $G(t,s)$ is given by (2.4),

$$C=\frac{\delta}{\delta-\sum_{j=1}^{m-2}b_j}\sum_{j=1}^{m-2}b_j\int_{\xi_j}^{\infty}p(s)f(s,u(s),u^{\Delta}(s))\Delta s$$

and

$$B=\frac{\sum_{j=1}^{m-2}a_j}{\delta-\sum_{j=1}^{m-2}b_j}\sum_{j=1}^{m-2}b_j\int_{\xi_j}^{\infty}p(s)f(s,u(s),u^{\Delta}(s))\Delta s+\sum_{j=1}^{m-2}a_j\int_{\xi_j}^{\infty}p(s)f(s,u(s),u^{\Delta}(s))\Delta s.$$

Proof. Since θ and φ are two linearly independent solutions of equation (2.1), we know that any solution of equation (1.1) can be represented by

$$u(t) = c_1\theta(t) + c_2\varphi(t) + \int_{t_1}^{\infty} G(t,s)p(s)f(s,u(s),u^{\Delta}(s))\Delta s,$$

where c_1 and c_2 are arbitrary constants and $G(t,s)$ is as in (2.4). If we make a direct calculation with boundary conditions of (1.1), we obtain

$$c_1 = \frac{1}{\left(\delta - \sum_{j=1}^{m-2} b_j\right)D} \sum_{j=1}^{m-2} b_j \int_{\xi_j}^{\infty} \varphi(s)p(s)f(s,u(s),u^{\Delta}(s))\Delta s$$

and

$$c_2 = \frac{\sum_{j=1}^{m-2} a_j}{\delta\left(\delta - \sum_{j=1}^{m-2} b_j\right)D} \sum_{j=1}^{m-2} b_j \int_{\xi_j}^{\infty} \varphi(s)p(s)f(s,u(s),u^{\Delta}(s))\Delta s$$
$$+ \frac{1}{\delta D} \sum_{j=1}^{m-2} a_j \int_{\xi_j}^{\infty} p(s)f(s,u(s),u^{\Delta}(s))\Delta s.$$

Hence, we find integral equation (2.5). This completes the proof.

Lemma 2.2. *Let*

$$K = \frac{\beta}{\alpha} + \int_{t_1}^{\infty} \frac{\Delta\tau}{p(\tau)} < \infty. \tag{2.6}$$

Then

$$0 < G(t,s) \le K$$

for $(t,s) \in [t_1,\infty)_{\mathbb{T}} \times [t_1,\infty)_{\mathbb{T}}$.

Proof. It is clear that $G(t,s) > 0$. Now we are in a position to show that $G(t,s) \le K$.

(i) For $s \in [t_1,\infty)_{\mathbb{T}}$ and $t \le s$, we get

$$G(t,s) = \frac{\beta}{\alpha} + \int_{t_1}^{t} \frac{\Delta\tau}{p(\tau)} \le \frac{\beta}{\alpha} + \int_{t_1}^{\infty} \frac{\Delta\tau}{p(\tau)} = K.$$

(ii) Letting $s \in [t_1,\infty)_{\mathbb{T}}$ and $t \ge s$, we have

$$G(t,s) = \frac{\beta}{\alpha} + \int_{t_1}^{s} \frac{\Delta\tau}{p(\tau)} \le \frac{\beta}{\alpha} + \int_{t_1}^{\infty} \frac{\Delta\tau}{p(\tau)} = K.$$

Thus, we obtain $0 < G(t,s) \le K$ for $(t,s) \in [t_1,\infty)_{\mathbb{T}} \times [t_1,\infty)_{\mathbb{T}}$. This completes the proof.

Lemma 2.3. *Green's function $G(t,s)$ in (2.4) and $\theta(t)$ in (2.3) satisfies*

$$G(t,s) \ge \gamma K$$

and

$$\theta(t) \ge \gamma K$$

for all $(t,s) \in [t_1,\infty)_{\mathbb{T}} \times [t_1,\infty)_{\mathbb{T}}$, where $\gamma = \min\left\{ \frac{\beta}{\beta+\alpha\int_{t_1}^{\infty}\frac{\Delta\tau}{p(\tau)}}, \frac{\beta}{\frac{\beta}{\alpha}+\int_{t_1}^{\infty}\frac{\Delta\tau}{p(\tau)}+\beta} \right\}$ and K is given by (2.6).

Proof.

(i) For $s \in [t_1,\infty)_{\mathbb{T}}$ and $s \le t$, we obtain that

$$G(t,s) = \frac{\beta}{\alpha} + \int_{t_1}^{t} \frac{\Delta\tau}{p(\tau)} \quad \ge \quad \frac{\beta}{\beta+\alpha\int_{t_1}^{\infty}\frac{\Delta\tau}{p(\tau)}} K \ge \gamma K.$$

(ii) Taking $s \in [t_1,\infty)_{\mathbb{T}}$ and $s \ge t$, we find that

$$G(t,s) = \frac{\beta}{\alpha} + \int_{t_1}^{s} \frac{\Delta\tau}{p(\tau)} \quad \ge \quad \frac{\beta}{\beta+\alpha\int_{t_1}^{\infty}\frac{\Delta\tau}{p(\tau)}} K \ge \gamma K.$$

(iii) Since

$$0 < \frac{\frac{\beta}{\alpha}+\int_{t_1}^{\infty}\frac{\Delta\tau}{p(\tau)}}{\frac{\beta}{\alpha}+\int_{t_1}^{\infty}\frac{\Delta\tau}{p(\tau)}+\beta} < 1,$$

we have

$$\theta(t) = \beta + \alpha\int_{t_1}^{t} \frac{\Delta\tau}{p(\tau)} \quad \ge \quad \frac{\beta}{\frac{\beta}{\alpha}+\int_{t_1}^{\infty}\frac{\Delta\tau}{p(\tau)}+\beta} K \ge \gamma K.$$

Thus, the proof is completed.

Solving the BVP (1.1) is equivalent to finding fixed points of operator A defined by

$$Au(t) = \int_{t_1}^{\infty} G(t,s)p(s)f(s,u(s),u^{\Delta}(s))\Delta s + \frac{C}{D}\theta(t) + \frac{B}{D}\varphi(t)$$

for $t \in [t_1,\infty)_{\mathbb{T}}$. Let $\mathbb{B}$ be a Banach space defined by

$$\mathbb{B} = \left\{ u \in C^{\Delta}[t_1,\infty)_{\mathbb{T}} : \sup_{t\in[t_1,\infty)_{\mathbb{T}}} \frac{|u(t)|}{1+t} < \infty, \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |u^{\Delta}(t)| < \infty \right\}$$

with norm $\|u\| = \max\{\|u\|_1, \|u^{\Delta}\|_{\infty}\}$, where

$$\|u\|_1 = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} \frac{|u(t)|}{1+t}, \|u^{\Delta}\|_{\infty} = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |u^{\Delta}(t)|$$

and defined cone $P \subset \mathbb{B}$ by

$$P = \left\{ u \in \mathbb{B} : \alpha u(t_1) - \beta p(t_1) u^{\Delta}(t_1) = \sum_{j=1}^{m-2} a_j p(\xi_j) u^{\Delta}(\xi_j) \text{ and } u \text{ is nonnegative on } [t_1,\infty)_{\mathbb{T}} \right\}. \tag{2.7}$$

Lemma 2.4. *If $u \in P$, then we have $\|u\|_1 \le M\|u^{\Delta}\|_{\infty}$, where*

$$M = \max\left\{ \frac{\beta}{\alpha} p(t_1) - t_1 + \frac{\sum_{j=1}^{m-2} a_j p(\xi_j)}{\alpha}, 1 \right\}.$$

Proof. Since $u \in P$, we find that

$$\begin{aligned} \frac{u(t)}{1+t} &= \frac{1}{1+t}\left(\int_{t_1}^{t} u^{\Delta}(\tau)\Delta\tau + \frac{\beta}{\alpha} p(t_1) u^{\Delta}(t_1) + \frac{1}{\alpha}\sum_{j=1}^{m-2} a_j p(\xi_j) u^{\Delta}(\xi_j) \right) \\ &\le \left(\frac{t - t_1 + \frac{\beta}{\alpha} p(t_1) + \frac{1}{\alpha}\sum_{j=1}^{m-2} a_j p(\xi_j)}{1+t} \right) \|u^{\Delta}\|_{\infty} \le M\|u^{\Delta}\|_{\infty} \end{aligned}$$

for all $t \in [t_1,\infty)_{\mathbb{T}}$. Hence, the proof is complete.

Remark 2.5. Since $\|u\| = \max\{\|u\|_1, \|u^{\Delta}\|_{\infty}\}$, we have $\|u\| = \|u\|_1$ or $\|u\| = \|u^{\Delta}\|_{\infty}$. If $\|u\| = \|u\|_1$, then we get from Lemma 2.4 that $\|u\| \le M\|u^{\Delta}\|_{\infty}$. If $\|u\| = \|u^{\Delta}\|_{\infty}$, then we find $\|u\| \le M\|u^{\Delta}\|_{\infty}$ by using $M \ge 1$. Hence, we obtain $\|u\| \le M\|u^{\Delta}\|_{\infty}$ for all $u \in P$.

Lemma 2.6. *$A : P \to P$ is a completely continuous operator.*

Proof. It is clear that $A(P) \subset P$. We divide the proof into two steps.

Step 1. We show that $A : P \to P$ is continuous.

Let $u_n \to u$ as $n \to \infty$ in P. Then there exists $r_0 > 0$ such that $\sup_{n\in\mathbb{N}} \|u_n\| < r_0$. Since

$$S_{r_0} := \sup\{h((1+t)x, y) : 0 \le x \le r_0, 0 \le y \le r_0\} < \infty,$$

we find from $(H1)$ that

$$\int_{t_1}^{\infty} p(s)|f(s,u_n,u_n^{\Delta})-f(s,u,u^{\Delta})|\Delta s \leq 2S_{r_0}\int_{t_1}^{\infty} p(s)\vartheta(s)\Delta s < \infty.$$

A direct calculation gives

$$\begin{aligned}(Au)^{\Delta}(t) = \frac{1}{p(t)}\Bigg(&\int_{t}^{\infty} p(s)f(s,u(s),u^{\Delta}(s))\Delta s \\ &+\frac{1}{\delta-\sum\limits_{j=1}^{m-2} b_j}\sum_{j=1}^{m-2} b_j\int_{\xi_j}^{\infty} p(s)f(s,u(s),u^{\Delta}(s))\Delta s\Bigg).\end{aligned}$$

Using the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}|(Au_n)^{\Delta}(t)-(Au)^{\Delta}(t)| &\leq \frac{1}{p(t)}\Bigg(\int_{t_1}^{\infty} p(s)|f(s,u_n,u_n^{\Delta})-f(s,u,u^{\Delta})|\Delta s \\ &+ \frac{1}{\delta-\sum\limits_{j=1}^{m-2} b_j}\sum_{j=1}^{m-2} b_j\int_{\xi_j}^{\infty} p(s)|f(s,u_n,u_n^{\Delta})-f(s,u,u^{\Delta})|\Delta s\Bigg) \\ &\to 0\ (n\to\infty).\end{aligned}$$

Furthermore, $\|Au_n - Au\| \leq M\|(Au_n)^{\Delta}-(Au)^{\Delta}\|_{\infty} \to 0$ as $n\to\infty$. So, A is continuous.

Step 2. We show that the image of any bounded subset of P under A is relatively compact in P.

Let $\Omega \subset P$ be any bounded subset. Then there exists $r > 0$ such that $\|u\| \leq r$ for all $u \in \Omega$. From $(H1)$, we obtain

$$S_r = \sup\left\{h(u(s),u^{\Delta}(s)) : 0 \leq \frac{u(s)}{1+s} \leq r,\ 0 \leq u^{\Delta}(s) \leq r\right\} < \infty.$$

For $\forall u \in \Omega$, we find from $(H1)$ that

$$\begin{aligned}
\|Au\| &\leq M\|(Au)^{\Delta}\|_{\infty} \\
&\leq M \sup_{t\in[t_1,\infty)_{\mathbb{T}}} \left(\frac{1}{p(t)}\right)\left(\int_{t_1}^{\infty} p(s)f(s,u(s),u^{\Delta}(s))\Delta s\right. \\
&+ \left.\frac{1}{\delta-\sum\limits_{j=1}^{m-2} b_j}\sum_{j=1}^{m-2} b_j \int_{\xi_j}^{\infty} p(s)f(s,u(s),u^{\Delta}(s))\Delta s\right) \\
&\leq MS_r \sup_{t\in[t_1,\infty)_{\mathbb{T}}} \left(\frac{1}{p(t)}\right)\left(\int_{t_1}^{\infty} p(s)\vartheta(s)\Delta s + \frac{1}{\delta-\sum\limits_{j=1}^{m-2} b_j}\sum_{j=1}^{m-2} b_j \int_{\xi_j}^{\infty} p(s)\vartheta(s)\Delta s\right) \\
&< \infty.
\end{aligned}$$

Hence $A\Omega$ is uniformly bounded. Now, we show that $A\Omega$ is equicontinuous on $[t_1,\infty)_{\mathbb{T}}$. For any $R>0$, $t_1^*, t_2^* \in [0,R]_{\mathbb{T}}$ and for all $u\in\Omega$, we may, without loss of generality, assume that $t_2^* > t_1^*$. By using $(H1)$, we have

$$\begin{aligned}
&|(Au)^{\Delta}(t_2^*) - (Au)^{\Delta}(t_1^*)| \\
&= \left|\frac{1}{p(t_2^*)}\left(\int_{t_2^*}^{\infty} p(s)f(s,u_n(s),u_n^{\Delta}(s))\Delta s + \frac{1}{\delta-\sum\limits_{j=1}^{m-2} b_j}\sum_{j=1}^{m-2} b_j \int_{\xi_j}^{\infty} p(s)f(s,u_n(s),u_n^{\Delta}(s))\Delta s\right)\right. \\
&\quad \left. - \frac{1}{p(t_1^*)}\left(\int_{t_1^*}^{\infty} p(s)f(s,u(s),u^{\Delta}(s))\Delta s + \frac{1}{\delta-\sum\limits_{j=1}^{m-2} b_j}\sum_{j=1}^{m-2} b_j \int_{\xi_j}^{\infty} p(s)f(s,u(s),u^{\Delta}(s))\Delta s\right)\right| \\
&\leq \left|\frac{1}{p(t_2^*)} - \frac{1}{p(t_1^*)}\right| \frac{S_r}{\delta-\sum\limits_{j=1}^{m-2} b_j}\sum_{j=1}^{m-2} b_j \int_{\xi_j}^{\infty} p(s)\vartheta(s)\Delta s \\
&\quad + \left|\frac{1}{p(t_2^*)} - \frac{1}{p(t_1^*)}\right| S_r \int_{t_2^*}^{\infty} p(s)\vartheta(s)\Delta s + \frac{1}{p(t_1^*)} S_r \int_{t_1^*}^{t_2^*} p(s)\vartheta(s)\Delta s \\
&\to 0
\end{aligned}$$

uniformly as $t_1^* \to t_2^*$. So, we get $\|(Au)^{\Delta}(t_2^*) - (Au)^{\Delta}(t_1^*)\|_{\infty} \to 0$ uniformly as $t_1^* \to t_2^*$. Since $\|(Au)(t_2^*) - (Au)(t_1^*)\| \leq \|(Au)^{\Delta}(t_2^*) - (Au)^{\Delta}(t_1^*)\|_{\infty}$, we obtain $\|(Au)(t_2^*) - (Au)(t_1^*)\| \to 0$ uniformly as $t_1^* \to t_2^*$. Therefore $A\Omega$ is equicontinuous on any compact interval of $[t_1,\infty)_{\mathbb{T}}$.

Now, we show that $A\Omega$ is equiconvergent on $[t_1,\infty)_{\mathbb{T}}$. For any $u\in\Omega$, we have

$$\begin{aligned}
\lim_{t\to\infty}\left|\frac{Au(t)}{1+t}\right| &= \lim_{t\to\infty}\frac{1}{1+t}\left|\int_{t_1}^{\infty}G(t,s)p(s)f(s,u(s),u^{\Delta}(s))\Delta s+\frac{C}{D}\theta(t)+\frac{B}{D}\varphi(t)\right| \\
&\leq \lim_{t\to\infty}\frac{1}{1+t}\left(KS_r\int_{t_1}^{\infty}p(s)\vartheta(s)\Delta s\right) \\
&\quad +\lim_{t\to\infty}\frac{1}{1+t}\frac{C}{D}\left(\beta+\alpha\int_{t_1}^{\infty}\frac{\Delta\tau}{p(\tau)}\right)+\lim_{t\to\infty}\frac{1}{1+t}\frac{B}{D}\delta \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
|(Au)^{\Delta}(t)-(Au)^{\Delta}(\infty)| \leq &\left|\frac{1}{p(t)}-\frac{1}{p(\infty)}\right|\frac{S_r}{\delta-\sum\limits_{j=1}^{m-2}b_j}\sum_{j=1}^{m-2}b_j\int_{\xi_j}^{\infty}p(s)\vartheta(s)\Delta s \\
&+\frac{1}{p(t)}S_r\int_{t}^{\infty}p(s)\vartheta(s)\Delta s \\
&\to 0
\end{aligned}$$

as $t\to\infty$. So, $A\Omega$ is equiconvergent at infinity. As a consequence of this steps, we get $A:P\to P$ is completely continuous.

3. Main results

To prove the existence of least one positive solution for the BVP (1.1), we will use the following Leray-Schauder fixed point theorem.

Theorem 3.1. [24] *Let E be a real Banach space, and $A:E\to E$ be a completely continuous operator. If*

$$\left\{x\in E: x=\lambda Ax,\ 0<\lambda<1\right\}$$

is bounded, then A has a fixed point in the closed set $T\subset E$, where

$$T=\left\{x\in E:\|x\|\leq R\right\},\quad R=\sup\left\{\|x\|: x=\lambda Ax,\ 0<\lambda<1\right\}.$$

Theorem 3.2. *The BVP (1.1) has at least one positive solution.*

Proof. Define the cone P as in (2.7). From Lemma 2.6, we see that $A : P \to P$ is completely continuous operator. We denote

$$N(A) = \{u \in P : u = \lambda Au, 0 < \lambda < 1\}.$$

Now we show that the set $N(A)$ is bounded. Let

$$T = \{u \in P : \|u\| \leq R\},$$

where $R = \sup\{\|u\| : u = \lambda Au, 0 < \lambda < 1\}$. If $\|u\| \leq R$, then $0 \leq \frac{u(t)}{1+t} \leq R$ and $0 \leq u^{\Delta}(t) \leq R$ for $t \in [t_1, \infty)_{\mathbb{T}}$. It follows that

$$S_R = \sup\left\{h(u(t), u^{\Delta}(t)) : 0 \leq \frac{u(t)}{1+t} \leq R, 0 \leq u^{\Delta}(t) \leq R\right\} < \infty, \quad \forall t \in [t_1, \infty)_{\mathbb{T}}.$$

For all $u \in N(A)$, we obtain from $(H1)$ that

$$\begin{aligned}
\|u\| &\leq \lambda M \|(Au)^{\Delta}\|_{\infty} \\
&\leq \lambda M \sup_{t \in [t_1, \infty)_{\mathbb{T}}} \left(\frac{1}{p(t)}\right) \Bigg(\int_{t_1}^{\infty} p(s)\vartheta(s) h(u(s), u^{\Delta}(s)) \Delta s \\
&\quad + \frac{1}{\delta - \sum\limits_{j=1}^{m-2} b_j} \sum_{j=1}^{m-2} b_j \int_{\xi_j}^{\infty} p(s)\vartheta(s) h(u(s), u^{\Delta}(s)) \Delta s \Bigg) \\
&\leq \lambda M S_R \sup_{t \in [t_1, \infty)_{\mathbb{T}}} \left(\frac{1}{p(t)}\right) \left(\int_{t_1}^{\infty} p(s)\vartheta(s) \Delta s + \frac{1}{\delta - \sum\limits_{j=1}^{m-2} b_j} \sum_{j=1}^{m-2} b_j \int_{\xi_j}^{\infty} p(s)\vartheta(s) \Delta s \right) \\
&< \infty.
\end{aligned}$$

Thus, $N(A)$ is a bounded set. By Theorem 3.1, the BVP (1.1) has at least one positive solution. This completes the proof.

Now, we are in a position to present the five functionals fixed point theorem. Let γ, β, θ be nonnegative continuous convex functionals on the cone P and let α, ψ be nonnegative continuous concave functionals on cone P. For nonnegative numbers l, a, b, d and c, define the following convex sets:

$$
\begin{aligned}
P(\gamma,c) &= \{x \in P : \gamma(x) < c\},\\
P(\gamma,\alpha,a,c) &= \{x \in P : a \le \alpha(x), \gamma(x) \le c\},\\
Q(\gamma,\beta,d,c) &= \{x \in P : \beta(x) \le d, \gamma(x) \le c\}, \qquad (3.1)\\
P(\gamma,\theta,\alpha,a,b,c) &= \{x \in P : a \le \alpha(x), \theta(x) \le b, \gamma(x) \le c\},\\
Q(\gamma,\beta,\Psi,l,d,c) &= \{x \in P : l \le \Psi(x), \beta(x) \le d, \gamma(x) \le c\}.
\end{aligned}
$$

Theorem 3.3. (Five Functionals Fixed Point Theorem [2]) *Let P be a cone in a real Banach space E. Suppose that there exist nonnegative numbers c and r, nonnegative continuous concave functionals α and Ψ on P, and nonnegative continuous convex functionals γ, ϕ and θ on P, with*

$$\alpha(x) \le \phi(x), \|x\| \le r\gamma(x), \forall x \in \overline{P(\gamma,c)}.$$

Suppose that $A : \overline{P(\gamma,c)} \to \overline{P(\gamma,c)}$ is a completely continuous and there exist nonnegative numbers b,d,m,l, with $0 < d < m$ such that

(i) *$\{x \in P(\gamma,\theta,\alpha,m,b,c) : \alpha(x) > m\} \neq \emptyset$ and $\alpha(Ax) > m$ for $x \in P(\gamma,\theta,\alpha,m,b,c)$,*
(ii) *$\{x \in Q(\gamma,\phi,\Psi,l,d,c) : \phi(x) < d\} \neq \emptyset$ and $\phi(Ax) < d$ for $x \in Q(\gamma,\phi,\Psi,l,d,c)$,*
(iii) *$\alpha(Ax) > b$, for $x \in P(\gamma,\alpha,m,c)$, with $\theta(Ax) > b$,*
(iv) *$\phi(Ax) < d$, for $x \in Q(\gamma,\phi,d,c)$. with $\Psi(Ax) < l$,*

Then A has at least three fixed points $x_1,x_2,x_3 \in \overline{P(\gamma,c)}$ such that

$$\phi(x_1) < d, \quad \alpha(x_2) > m, \quad \phi(x_3) > d \quad \text{with } \alpha(x_3) < m.$$

Let $1 < k < \infty$, $\frac{1}{k} \in \mathbb{T}$ be fixed and let $l = 0$, $r = 1$. Define the nonnegative continuous concave functionals ϕ, ψ and the nonnegative continuous convex functionals ρ, λ, θ' on P by

$$\phi(u) = \frac{k}{1+k} \min_{t \in [\frac{1}{k},\infty)_{\mathbb{T}}} u(t),\ \rho(u) = \lambda(u) = \theta'(u) = \|u\|,\ \psi(u) \equiv 0.$$

In addition, we have $\phi(u) \le \lambda(u)$ and $\|u\| \le r\rho(u)$ for $u \in P$. Define the constants,

$$x = \int_{\frac{1}{k}}^{k} p(s)\omega(s)\Delta s + \frac{1}{\alpha\left(\delta - \sum\limits_{j=1}^{m-2} b_j\right)} \sum_{j=1}^{m-2} b_j \int_{\frac{1}{k}}^{k} p(s)\omega(s)\Delta s$$

and

$$y=\sup_{t\in[t_1,\infty)_{\mathbb{T}}}\left(\frac{1}{p(t)}\right)\left(\int_{t_1}^{\infty}p(s)\vartheta(s)\Delta s+\frac{1}{\delta-\sum_{j=1}^{m-2}b_j}\sum_{j=1}^{m-2}b_j\int_{\xi_j}^{\infty}p(s)\vartheta(s)\Delta s\right).$$

To state this main result, we need the following assumption:

(H2) For all $t\in[t_1,\infty)_{\mathbb{T}}$, $\omega(t)h((1+t)x,y)\leq f(t,(1+t)x,y)$, where $\omega:[t_1,\infty)_{\mathbb{T}}\to(0,\infty)$ is continuous and $\int_{t_1}^{\infty}\omega(s)p(s)\Delta(s)<\infty$.

Now, we use the five functionals fixed point theorem to prove the next theorem.

Theorem 3.4. *Let $\xi_{m-2}\leq\frac{1}{k}$ and $\frac{1}{k}\in\mathbb{T}$ for each $k\in\{1,2,...,n\}$. Assume that $(H2)$ holds. Suppose that there exist positive numbers $0<d<a<c$ such that,*

(i) $h((1+t)u,v)\leq\frac{c}{My}$ for $(t,u,v)\in[t_1,\infty)_{\mathbb{T}}\times[0,c]\times[0,c]$,
(ii) $h((1+t)u,v)>\frac{a\alpha(1+k)}{\gamma\beta kx}$ for $(t,u,v)\in[\frac{1}{k},k]_{\mathbb{T}}\times[\frac{a}{k},c]\times[0,c]$,
(iii) $h((1+t)u,v)<\frac{d}{My}$ for $(t,u,v)\in[t_1,\infty)_{\mathbb{T}}\times[0,d]\times[0,d]$.

Then the BVP (1.1) has at least three positive solutions u_1, u_2 and u_3 satisfying $\|u_1\|<d$, $\phi(u_2)>a$, $d<\|u_3\|$ with $\phi(u_3)<a$.

Proof. First, we show that $A:\overline{P(\rho,c)}\to\overline{P(\rho,c)}$. If $u\in\overline{P(\rho,c)}$, then $\rho(u)\leq c$. This implies $0\leq\frac{u(t)}{1+t}\leq c$, $0\leq u^{\Delta}(t)\leq c$ for $t\in[t_1,\infty)_{\mathbb{T}}$. By $(H1)$ and (i), we obtain

$$\begin{aligned}\rho(Au)&\leq M\|(Au)^{\Delta}\|_{\infty}\\&\leq M\sup_{t\in[t_1,\infty)_{\mathbb{T}}}\left(\frac{1}{p(t)}\right)\left(\int_{t_1}^{\infty}p(s)\vartheta(s)h(u(s),u^{\Delta}(s))\Delta s\right.\\&\quad\left.+\frac{1}{\delta-\sum_{j=1}^{m-2}b_j}\sum_{j=1}^{m-2}b_j\int_{\xi_j}^{\infty}p(s)\vartheta(s)h(u(s),u^{\Delta}(s))\Delta s\right)\\&\leq M\frac{c}{My}\sup_{t\in[t_1,\infty)_{\mathbb{T}}}\left(\frac{1}{p(t)}\right)\left(\int_{t_1}^{\infty}p(s)\vartheta(s)\Delta s+\frac{1}{\delta-\sum_{j=1}^{m-2}b_j}\sum_{j=1}^{m-2}b_j\int_{\xi_j}^{\infty}p(s)\vartheta(s)\Delta s\right)=c.\end{aligned}$$

Therefore $A:\overline{P(\rho,c)}\to\overline{P(\rho,c)}$. From Lemma , $A:\overline{P(\rho,c)}\to\overline{P(\rho,c)}$ is completely continuous.

Next, we show that the conditions of Theorem 3.3 is satisfied with $b=c$. We take $u(t)=\frac{c+a}{2}(t+1)$ for $t\in[t_1,\infty)_{\mathbb{T}}$. It is easy to see that $u(t)\in P$ and

$$\phi(u)=\frac{k}{1+k}\frac{c+a}{2}\left(\frac{1}{k}+1\right)>a,$$

$$\theta'(u) = \frac{c+a}{2} = \frac{b+a}{2} < b, \rho(u) = \frac{c+a}{2} < c,$$

that is, $\{u \in P(\rho, \theta', \phi, a, b, c) : \phi(u) > a\} \neq \emptyset$. If $u \in P(\rho, \theta', \phi, a, b, c)$, then $\frac{a}{k} \leq \frac{u(t)}{1+t} \leq c$ and $0 \leq u^{\Delta}(t) \leq c$ for $t \in [\frac{1}{k}, k]_{\mathbb{T}}$. By Lemma 2.3, $(H2)$ and condition (ii), we have

$$\begin{aligned}
\phi(Au) &= \frac{k}{1+k} \min_{t \in [\frac{1}{k}, \infty)_{\mathbb{T}}} \left(\int_{t_1}^{\infty} G(t,s) p(s) f(s, u(s), u^{\Delta}(s)) \Delta s + \frac{C}{D} \theta(t) + \frac{B}{D} \varphi(t) \right) \\
&\geq \frac{k}{1+k} \left(\int_{t_1}^{\infty} \gamma K p(s) f(s, u(s), u^{\Delta}(s)) \Delta s + \frac{C}{D} \gamma K + \frac{B}{D} \delta \right) \\
&> \frac{k}{1+k} \left(\int_{t_1}^{\infty} \gamma \left(\frac{\beta}{\alpha} + \int_{t_1}^{\infty} \frac{\Delta \tau}{p(\tau)} \right) p(s) f(s, u(s), u^{\Delta}(s)) \Delta s + \frac{C}{D} \gamma \left(\frac{\beta}{\alpha} + \int_{t_1}^{\infty} \frac{\Delta \tau}{p(\tau)} \right) \right) \\
&> \gamma \frac{\beta}{\alpha} \frac{k}{1+k} \left(\int_{t_1}^{\infty} p(s) f(s, u(s), u^{\Delta}(s)) \Delta s \right. \\
&\quad \left. + \frac{1}{\alpha \left(\delta - \sum_{j=1}^{m-2} b_j \right)} \sum_{j=1}^{m-2} b_j \int_{\xi_j}^{\infty} p(s) f(s, u(s), u^{\Delta}(s)) \Delta s \right) \\
&\geq \gamma \frac{\beta}{\alpha} \frac{k}{1+k} \left(\int_{t_1}^{\infty} p(s) \omega(s) h(u(s), u^{\Delta}(s)) \Delta s \right. \\
&\quad \left. + \frac{1}{\alpha \left(\delta - \sum_{j=1}^{m-2} b_j \right)} \sum_{j=1}^{m-2} b_j \int_{\xi_j}^{\infty} p(s) \omega(s) h(u(s), u^{\Delta}(s)) \Delta s \right) \\
&\geq \gamma \frac{\beta}{\alpha} \frac{k}{1+k} \left(\int_{\frac{1}{k}}^{k} p(s) \omega(s) h(u(s), u^{\Delta}(s)) \Delta s \right. \\
&\quad \left. + \frac{1}{\alpha \left(\delta - \sum_{j=1}^{m-2} b_j \right)} \sum_{j=1}^{m-2} b_j \int_{\frac{1}{k}}^{k} p(s) \omega(s) h(u(s), u^{\Delta}(s)) \Delta s \right) \\
&> \gamma \frac{\beta}{\alpha} \frac{k}{1+k} \frac{a\alpha(1+k)}{\gamma \beta k x} \left(\int_{\frac{1}{k}}^{k} p(s) \omega(s) \Delta s + \frac{1}{\alpha \left(\delta - \sum_{j=1}^{m-2} b_j \right)} \sum_{j=1}^{m-2} b_j \int_{\frac{1}{k}}^{k} p(s) \omega(s) \Delta s \right) \\
&= a.
\end{aligned}$$

So, we obtain

$$\phi(Au) > a. \tag{3.2}$$

Hence, condition (i) of Theorem 3.3 holds. We take $u(t) = \frac{d}{2}(1+t)$ for $\forall t \in [\frac{1}{k}, \infty)_{\mathbb{T}}$. It is easy to see that $u(t) \in P$ and

$$\begin{aligned} \psi(u) &= 0 \geq l = 0, \\ \lambda(u) &= \frac{d}{2} < d, \\ \rho(u) &= \frac{d}{2} < c, \end{aligned}$$

that is, $\{u \in Q(\rho, \lambda, \psi, l, d, c) : \lambda(u) < d\} \neq \emptyset$. By $(H1)$ and condition (iii), we get, for $u \in Q(\rho, \lambda, \psi, l, d, c)$, that

$$\begin{aligned} \lambda(Au) &= \|Au\| \\ &\leq M\|(Au)^{\Delta}\|_{\infty} \\ &\leq M \sup_{t\in[t_1,\infty)_{\mathbb{T}}} \left(\frac{1}{p(t)}\right)\left(\int_{t_1}^{\infty} p(s)\vartheta(s)h(u(s),u^{\Delta}(s))\Delta s\right. \\ &\quad \left. + \frac{1}{\delta - \sum\limits_{j=1}^{m-2} b_j} \sum_{j=1}^{m-2} b_j \int_{\xi_j}^{\infty} p(s)\vartheta(s)h(u(s),u^{\Delta}(s))\Delta s\right) \\ &< M\frac{d}{My} \sup_{t\in[t_1,\infty)_{\mathbb{T}}} \left(\frac{1}{p(t)}\right)\left(\int_{t_1}^{\infty} p(s)\vartheta(s)\Delta s + \frac{1}{\delta - \sum\limits_{j=1}^{m-2} b_j} \sum_{j=1}^{m-2} b_j \int_{\xi_j}^{\infty} p(s)\vartheta(s)\Delta s\right) \\ &= d. \end{aligned}$$

Thus, condition (ii) of Theorem 3.3 is satisfied. Since

$$P(\rho, \phi, a, c) = \{u \in P : \frac{k}{1+k} \min_{t\in[\frac{1}{k},\infty)_{\mathbb{T}}} u(t) \geq a, \|u\| \leq c\},$$

we get $\phi(Au) > a$ for $u \in P(\rho, \phi, a, c)$ according to (3.2). Therefore (iii) of Theorem 3.3 is satisfied. Finally, we shall verify that the condition (iv) of Theorem 3.3 holds. Since $\psi(Au) < l = 0$ is impossible, we omit the condition (iv) of Theorem 3.3. Since all the conditions of Theorem 3.3 are satisfied, we see that the BVP (1.1) has at least three positive solutions u_1, u_2

and u_3 satisfying

$$\|u_1\| < d,\ \phi(u_2) > a,$$
$$d < \|u_3\| \text{ with } \phi(u_3) < a.$$

This completes the proof.

Example 3.5. Let $\mathbb{T}=\mathbb{R}$. Taking $m=3, t_1=0, \xi_1=\frac{1}{3}, \alpha=4, \beta=1, \delta=3, a_1=b_1=1, p(t)=e^t, k=2$, we consider the following boundary value problem:

$$\begin{cases} \frac{1}{e^t}(e^t u')' + f(t,u(t),u'(t)) = 0,\ t\in[0,\infty), \\ 4u(0)-u'(0) = e^{-\frac{1}{3}}u'(\frac{1}{3}), \\ 3\lim_{t\to\infty} e^t u'(t) = e^{-\frac{1}{3}}u'(\frac{1}{3}), \end{cases} \tag{3.3}$$

where

$$f(t,(1+t)u,v) = e^{-2t}\begin{cases} 1000u^6 + \frac{v}{2000},\ u\le 1, v>0 \\ 1000+\frac{v}{2000},\ u>1, v>0. \end{cases}$$

Choose $\omega(t)=\vartheta(t)=e^{-2t}$ and

$$h((1+t)u,v) = \begin{cases} 1000u^6 + \frac{v}{2000},\ u\le 1, v>0, \\ 1000+\frac{v}{2000},\ u>1, v>0. \end{cases}$$

By simple calculations, we obtain that $M=1, x=\frac{9}{8}(e^{-\frac{1}{2}}-e^{-2}), y=1+\frac{1}{2}e^{-\frac{1}{3}}, \gamma=0.2$. If we take $d=0.1, a=5, c=2000$, then all conditions in Theorem 3.4 are verified. Thus the BVP (3.3) has at least three positive solutions u_1, u_2 and u_3 satisfying

$$\|u_1\| < 0.1,\ \frac{2}{3}\min_{t\in[\frac{1}{2},\infty)}(u_2) > 5,$$
$$0.1 < \|u_3\| \text{ with } \frac{2}{3}\min_{t\in[\frac{1}{2},\infty)}(u_3) < 5.$$

Example 3.6. Let $\mathbb{T}=[\frac{1}{10},3]\cup[8,\infty)$. Taking $p(t)=t\sigma(t), m=4, t_1=\frac{1}{10}, \xi_1=\frac{1}{3}, \xi_2=\frac{1}{2}, \alpha=1, \beta=2, \delta=1, a_1=\frac{1}{9}, a_2=\frac{1}{4}, b_1=b_2=0, k=2$, we consider the following boundary value problem:

$$\begin{cases} \frac{1}{t\sigma(t)}(t\sigma(t)u^\Delta(t))^\Delta + f(t,u(t),u^\Delta(t)) = 0,\ t\in[\frac{1}{10},\infty), \\ u(\frac{1}{10})-2u^\Delta(\frac{1}{10}) = u^\Delta(\frac{1}{3})+u^\Delta(\frac{1}{2}), \\ \lim_{t\to\infty} t\sigma(t)u^\Delta(t) = 0, \end{cases} \tag{3.4}$$

where

$$f(t,(1+t)u,v)=\frac{1}{t^4}\begin{cases} 2.10^{-6}u^5+2.10^{-8}v,\ u\le 1,\ v>0,\\ 2.10^{-6}+2.10^{-8}v,\ u>1,\ v>0.\end{cases}$$

Choosing $\omega(t)=\vartheta(t)=\frac{1}{t^4}$ and

$$h((1+t)u,v)=\begin{cases} 2.10^{-6}u^5+2.10^{-8}v,\ u\le 1,\ v>0,\\ 2.10^{-6}+2.10^{-8}v,\ u>1,\ v>0,\end{cases}$$

we obtain $M=201.9$, $x=\frac{3}{2}$, $y=1127.3\overline{148}$, $\gamma=\frac{1}{7}$. If we take $d=1$, $a=90$, $c=100$, then all conditions in Theorem 3.4 are fulfilled. Thus the BVP (3.4) has at least three positive solutions u_1, u_2 and u_3 satisfying

$$\|u_1\|<1,\ \frac{2}{3}\min_{t\in[\frac{1}{2},\infty)}(u_2)>90,$$

$$1<\|u_3\| \text{ with } \frac{2}{3}\min_{t\in[\frac{1}{2},\infty)}(u_3)<90.$$

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